Review of Broad-band Impedance Models

Workshop on Single Bunch Instabilities
and Broad-band Impedance Models

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BB Models

- SPEAR Impedance Model
- Broad-Band Resonator Model
- Improved BB Model for Periodic Structures
- Improved BB Model for a Cavity
- Second BB Model for Periodic Structures
- Second BB Model for a Cavity
- Composite Model
- Modified Inductance Model

Impedance Models

The coupling impedance consists of contributions of many components with different characteristics. Each contribution is a complicated function of frequency, usually with many resonant peaks below or around the beam pipe cut-off frequency, as well as a characteristic high frequency tail. Many detailed features depend critically on exact dimensions of structures which may even vary with temperature or other parameters.

Often these details are not required to predict beam stability. For single bunch stability only short-range wakes are needed, corresponding to smoothed impedances obtained by averaging, resulting in limited resolution of details of impedance function. Beyond frequency c/σ bunches not sensitive to effects of impedance.

Several BB impedance models have been constructed which describe beam-wall interaction over wide range of bunch characteristics (charge distribution, bunch length, displacement etc). From these one obtains corresponding wake potentials, effective impedances, loss factors. Free parameters should be determined from results of numerical calculations and checked with bench or machine measurements.

The SPEAR Impedance Model

Historically, first model was *SPEAR impedance*, obtained from measurements of parameters of short bunches in a particular machine. Frequency dependence of real part of longitudinal impedance fitted with simple power law ω^a. For short bunches, exponent was $a = -0.68 \approx 2/3$, from variation of bunch length with $\xi^{1/(2+a)}$. $\xi = \alpha e I_b / \nu_s^2 E$ is *'scaling parameter'*, α momentum compaction factor, e unit charge, I_b bunch current, ν_s synchrotron tune, E beam energy.

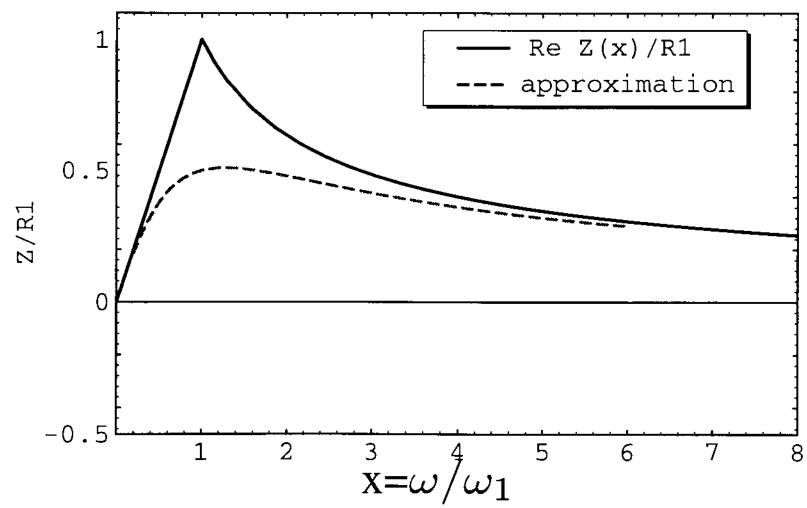

Figure 1: The $SPEAR$ impedance model.

The high frequency part of the (longitudinal) impedance is:

$$Z(\omega) = R_s \left(\frac{\omega}{\omega_1}\right)^a \quad \text{with} \quad a \approx -\frac{2}{3} \quad \text{for} \quad \omega > \omega_1 \,. \tag{1}$$

At low frequencies, however, the impedance is mainly inductive $Z = jR_s\omega/\omega_1$, the parameter a is close to unity. This yields the expected dependence of bunch length on current $\sigma \sim I_b^{1/3}$.

In addition to the resistance R_s one has to determine the frequency ω_1 where the two expressions change over. Measuring the dependence of the loss factor on bunch length at various energies and synchrotron tunes, it was found that R_s = 9 KΩ and $\omega_1 = 1.3$ GHz best describes the impedance of $SPEAR$, shown in Fig. 1. Although this model could be adjusted for other machines, it was often used indiscriminately with parameters valid only for $SPEAR$. The non-integer power law makes analytic calculations difficult.

The Broad-Band Resonator Model

More satisfying is the broad-band resonator model shown in Fig. 2, normalised by R_s.

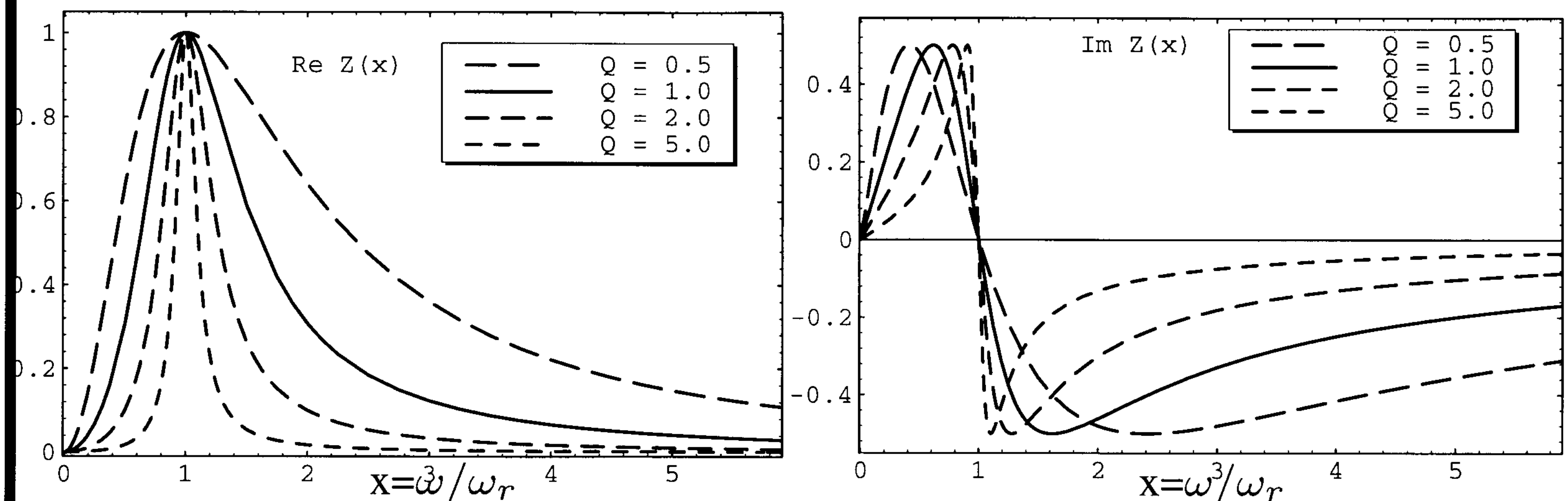

Figure 2: Normalised impedance of resonators with various quality factors.

It consists of a single expression, valid for all frequencies:

$$Z(\omega) = \frac{R_s}{1 + jQ_r(\omega/\omega_r - \omega_r/\omega)} \; . \tag{2}$$

3 parameters: shunt impedance R_s, resonant frequency ω_r and quality factor Q_r.

BB Resonator Model (2)

Main attraction of this model impedance: analytic evaluation of many expressions in closed form.

decompose quadratic denominator by partial fractions:

$$Z(\omega) = \frac{\omega_r \omega R/jQ}{\omega^2 - j\omega_r\omega/Q - \omega_r^2} = \frac{R}{jQ} \frac{\omega_r \omega}{(\omega - \omega_1)(\omega - \omega_2)} \tag{1}$$

$\omega_{1,2}$ complex roots of quadratic denominator. With modified quality factor $Q'_r = \sqrt{Q^2 - 1/4}$

$$\omega_{1,2} = \frac{\omega_r}{Q}\left(-\frac{j}{2} \pm Q'_r\right) . \tag{2}$$

$$Z(\omega) = \frac{R}{2jQ'_r}\left[\frac{\omega_1}{\omega - \omega_1} - \frac{\omega_2}{\omega - \omega_2}\right] , \tag{3}$$

Q'_r real for $Q > 1/2$, then $\omega_2 = -\omega_1^*$, For $Q < 1/2$: Q_r' imaginary, also roots of denominator purely imaginary:

$$\omega_{1,2} = -j\frac{\omega_r}{Q}\left(\frac{1}{2} \pm Q''\right) \;, \tag{4}$$

where $Q'' = \sqrt{1/4 - Q^2}$ real for $Q < 1/2$, but ω_2 no longer complex conjugate of $-\omega_1$.

Even simpler decomposition for "reduced impedance":

$$\frac{Z_{\|}}{n}(\omega) = \frac{\omega_0 R}{jQ_r'}\left[\frac{1}{\omega - \omega_1} - \frac{1}{\omega - \omega_2}\right] \;. \tag{5}$$

Similar decomposition for transverse impedance $Z_{\perp} \propto Z/n$.

Wake function: Fourier transform of impedance

$$W(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} Z(\omega) \exp j\omega\tau \ . \qquad (6)$$

For $\tau > 0$: close integration path in upper half plane of complex frequency, where $\exp(j\omega\tau) \to 0$ when $\tau \to \infty$, use residuum theorem:

$$W(\tau) = \frac{R}{2Q'_r} [\omega_1 \exp j\omega_1\tau - \omega_2 \exp j\omega_2\tau] \quad (7)$$

Wake function vanishes for $\tau < 0$ since contour must be closed in lower halfplane where impedance has no poles. For $\tau > 0$:

$$G(\tau) = \frac{\omega_r R_s}{Q} \mathrm{e}^{-\frac{\omega_r \tau}{2Q}} \left[\cos(\omega'_r \tau) - \frac{1}{2Q'_r} \sin(\omega'_r \tau)\right] \ , \qquad (8)$$

where $\omega'_r = \omega_r Q'_r / Q$. Transverse wake function even simpler, only a *sine* term.

The *reduced* longitudinal impedance Z/n, where $n = \omega/\omega_0$, proportional also to the *transverse impedance*, is shown in Fig. 3 normalised by $(Z/n)_0 = (\omega_0/\omega_r)(R_s/Q_r)$.

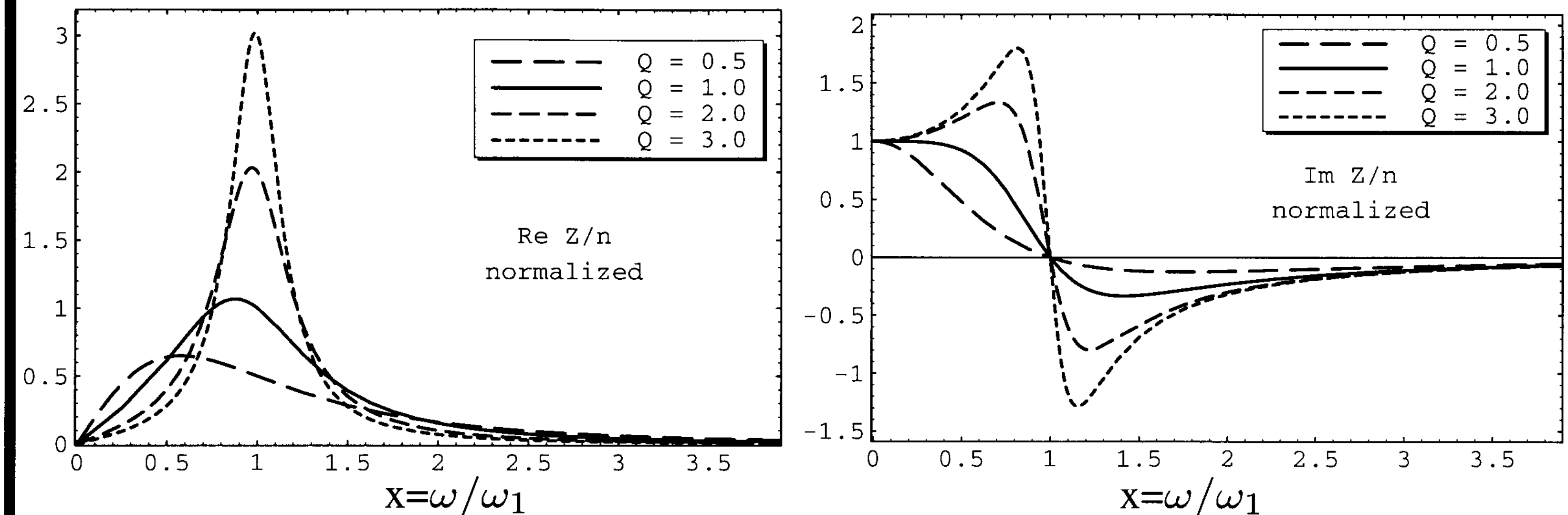

Figure 3: Reduced impedance of resonators with various quality factors.

At low frequencies, the real part of the resonator impedance depends quadratically on frequency, Re $Z \approx R_s\omega^2/Q_r^2\omega_r^2$, while the imaginary part is linear: Im $Z \approx R_s\omega/Q_r\omega_r$. The absolute value $|Z/n|$ approaches the constant value $(Z/n)_0$.

At high frequencies, the real part tends towards $\omega_r^2 R_s/\omega^2 Q_r^2$, i.e. it falls off with the inverse square of frequency (the imaginary part falls off only with the inverse first power). The fast decrease of the real part does not correspond to the expected asymptotic behaviour of a cavity impedance, whose real and imaginary parts fall only with the square root of frequency, nor to that of a periodic structure, where the real part should fall off with a 3/2 power. For very short bunches, for which the high-frequency part of the impedance is important, one may thus obtain too low estimates of stability limits and energy loss.

The main attraction of the resonator model impedance is that it permits analytic evaluation of many expressions in closed form, in particular since the quadratic denominator can be decomposed to linear ones by partial fractions. The longitudinal wake function becomes:

$$G(\tau) = \frac{\omega_r R_s}{Q_r} \left[\cos(\omega_r' \tau) - \frac{1}{2Q_r'} \sin(\omega_r' \tau) \right] \exp\left(-\frac{\omega_r \tau}{2Q_r} \right), \qquad (3)$$

where $\omega_r' = \omega_r Q_r'/Q_r$ and $Q_r' = \sqrt{Q_r^2 - 1/4}$. The transverse wake function is even simpler and has only a *sine* term, cf. Eq. (3.24). Both (normalized) wake functions are shown in Fig. 4.

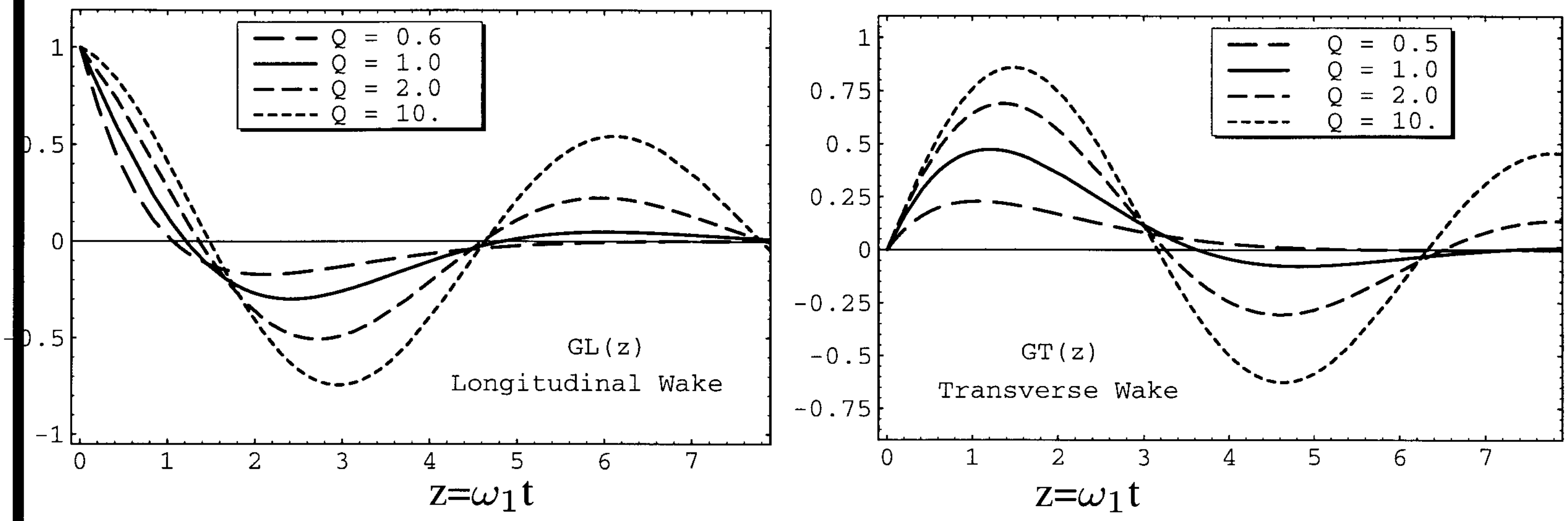

Figure 4: Normalised longitudinal and transverse wake functions of broad-band resonators with various quality factors.

This model also permits explicit evaluation of loss factors for Gaussian. However, the resulting decrease with the third power of the bunch length does not agree with experiment nor numerical calculations for cavities, where the loss factors fall exponentially. Hence several other impedance models were suggested.

Improved BB Model for Periodic Structures

There exist pairs of Fourier transforms with explicit expressions for the wake function in time domain, and the desired asymptotic dependence of the impedance in frequency domain. For a periodic structure, the real part of the longitudinal impedance decreases with frequency as $\omega^{-3/2}$. A suitable candidate for the wake function is the *complementary error function* erfc(z) = 1 - erf(z), with argument proportional to square root of time delay $z = \sqrt{\omega_1 \tau}$ (ω_1 free parameter). In order to make the real part vanish at zero frequency, we subtract a term to make the integral over all z equal to zero. An exponential will not change asymptotic behaviour of impedance, as it corresponds to an impedance whose real part decreases as ω^{-2} and thus vanishes faster than the main term. With a second free parameter ω_2, or rather the ratio $\alpha = \omega_2/\omega_1$, the wake function for "model 1A" can be written:

$$G(\tau) = \omega_1 R_s \left[2\,\mathrm{erfc}(\sqrt{\omega_1 \tau}) - \alpha \exp\left(-\alpha \sqrt{\omega_1 \tau}\right) \right] \tag{4}$$

for $\tau > 0$, while it vanishes for $\tau < 0$ as shown in Fig. 5, together with wake function for model 1B, for various values of the parameter α.

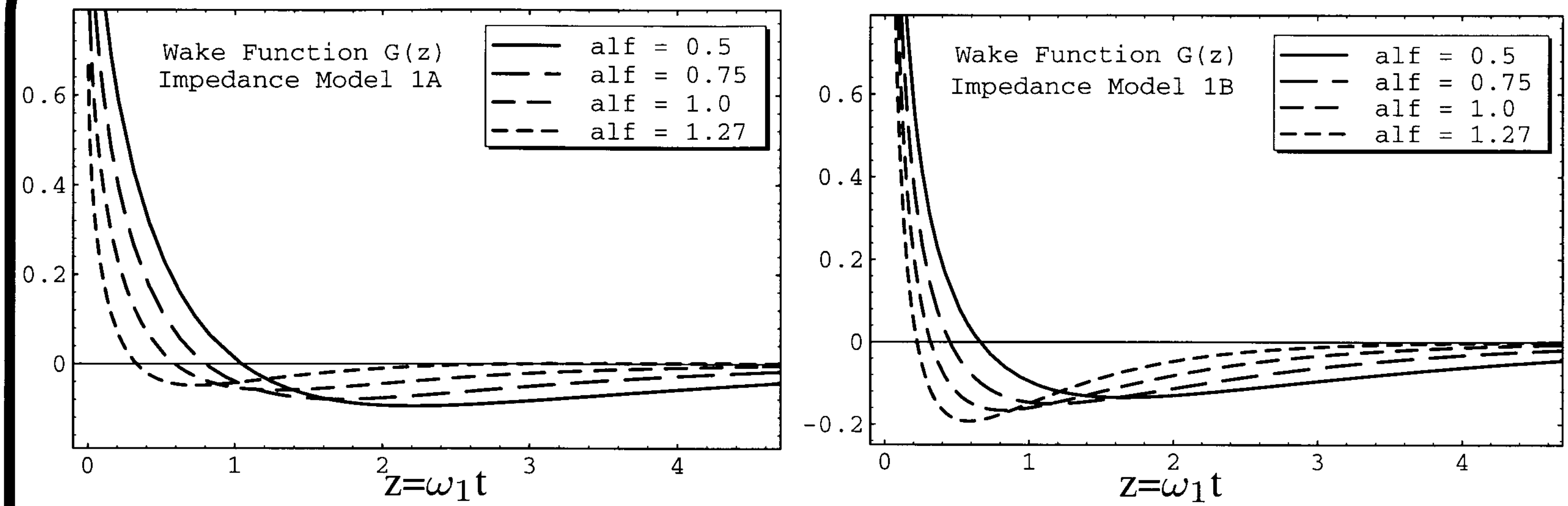

Figure 5: Normalised wake functions for model impedances 1A and 1B.

With $x = \omega/\omega_1$ and $u = \sqrt{x^2+1}$, the corresponding impedance is:

$$\begin{aligned}
\mathrm{Re}\, Z(x) &= R_s \left[\frac{1}{u} \sqrt{\frac{2}{u+1}} - \frac{\alpha^2}{x^2+\alpha^2} \right] , \\
\mathrm{Im} Z(x) &= R_s \left[\frac{1}{u} \sqrt{\frac{2}{u-1}} + \frac{\alpha^2}{x^2+\alpha^2} - \frac{2}{x} \right] , \qquad (5)
\end{aligned}$$

as shown in Fig. 6 for several values of $\alpha = \omega_2/\omega_1$.

The frequencies ω_1, ω_2 and the shunt impedance R_s are free parameters which can be chosen to fit the impedance of a particular structure.

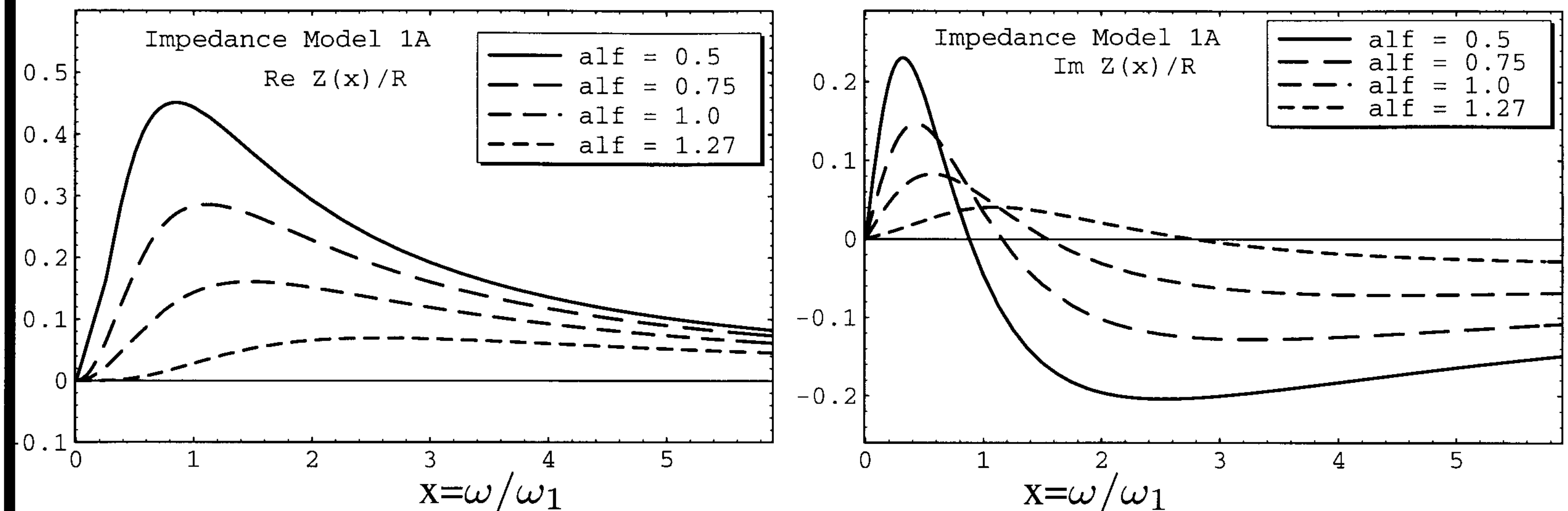

Figure 6: Real and imaginary parts for normalised impedance model 1A.

The asymptotic behaviour of the impedance is then given by:

$$\mathrm{Re}\, Z(x) \sim x^{-\frac{3}{2}}, \qquad \mathrm{Im}\, Z(x) \sim \frac{2-\alpha}{x}, \tag{6}$$

agrees with impedance of infinite periodic cavity array.

The expansion of the impedance at low frequencies is

$$\mathrm{Re}\, Z(x) \sim \left(\frac{1}{\alpha^2} - \frac{5}{8}\right) x^2 \,, \quad \mathrm{Im}\, Z(x) \sim \left(\frac{1}{\alpha^2} - \frac{3}{4}\right) x \,. \tag{7}$$

The impedance increases as ω^2, which gives a too slow decrease of loss factor for long Gaussian bunches, proportional to σ^{-3} as for resonator model. Improved by choosing $\alpha = \sqrt{5/8}$, for which value the impedance increases as ω^4 and the loss factor becomes proportional to σ^{-5}.

Improved BB Model for a Cavity

The expected asymptotic behaviour of the real part of the impedance of a cavity is $\omega^{-1/2}$. With $z = \sqrt{\omega_1 \tau}$, a suitable wake function for $\tau > 0$ is:

$$G(\tau) = R_s \left[\frac{\exp(-z)}{\sqrt{\pi z}} - \alpha \exp(-\alpha z) \right] . \tag{8}$$

With $x = \omega/\omega_1$ and $u = \sqrt{x^2 + 1}$ the corresponding impedance becomes:

$$\begin{aligned} \mathrm{Re}\, Z(x) &= R_s \left[\frac{\sqrt{u+1}}{\sqrt{2}u} - \frac{\alpha^2}{x^2 + \alpha^2} \right] , \\ \mathrm{Im}\, Z(x) &= -R_s \left[\frac{\sqrt{u-1}}{\sqrt{2}u} - \frac{\alpha x}{x^2 + \alpha^2} \right] . \end{aligned} \tag{9}$$

This impedance and its wake function, for several values of the parameter α, are shown in Fig. 7. The expansion of this impedance at low frequencies is

$$\mathrm{Re} Z(x) \sim \left(\frac{1}{\alpha^2} - \frac{3}{8} \right) x^2 , \quad \mathrm{Im} Z(x) \sim \left(\frac{1}{\alpha^2} - \frac{1}{2} \right) x . \tag{10}$$

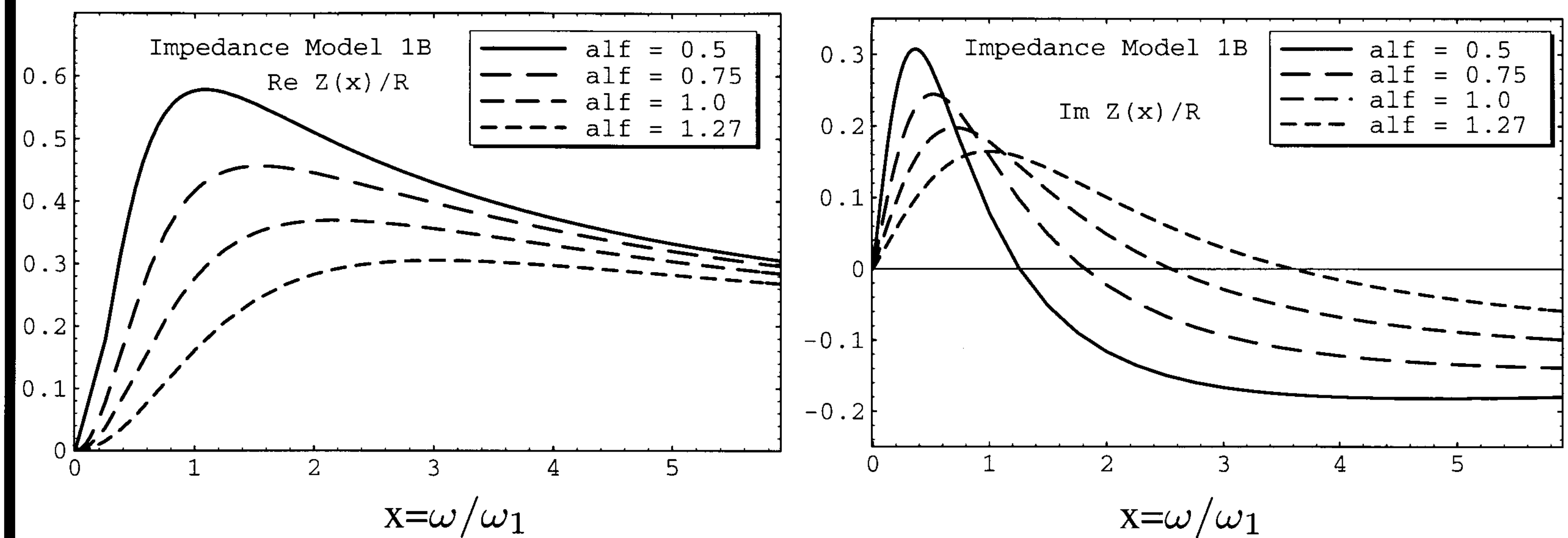

Figure 7: Real and imaginary parts of normalised impedance of model 1B.

The loss factor for a Gaussian bunch can be expressed analytically by parabolic cylinder functions or modified Bessel functions of order 1/4. Since these are rather unfamiliar, usually simpler to calculate it by numeric integration. Impedance and wake function for several values of parameter α are shown in Fig. 7.

Second BB Model for Periodic Structures

The models 1A and 1B describe asymptotic behaviour of impedance correctly, but give a too slow decrease of loss factor with bunch length for long Gaussian bunches. As shown in Fig. 8, the models 2A and 2B improve this by introducing a cut-off frequency ω_1 which occurs also for realistic structures. This impedance is no longer given by single expression for all frequencies.

For an asymptotic frequency dependence proportional to $\omega^{-3/2}$, such as for an infinite periodic array of cavities, one gets with $x = \omega/\omega_1$:

$$\mathrm{Re}\, Z(x) = R_s \begin{cases} 0 & \text{for } |x| < 1 \,, \\ \dfrac{\sqrt{|x|-1}}{x^2} & \text{for } |x| > 1 \,. \end{cases} \tag{11}$$

This function has a maximum at $x = \omega/\omega_1 = 4/3$, and decreases as $\omega^{-3/2}$ at high frequencies. Its Hilbert transform yields the imaginary part:

$$\operatorname{Im} Z(x) = \frac{R_s}{x^2} \begin{cases} \sqrt{1+x} - \sqrt{1-x} - x & \text{for } |x| < 1 \, , \\ \sqrt{1+x} - x & \text{for } |x| > 1 \, . \end{cases} \tag{12}$$

At low frequencies $x \ll 1$, the impedance is purely inductive Im $Z/R_s \approx 3x/4$ - the real part is identically zero. The corresponding wake function can be expressed by Fresnel functions, but is easier found by evaluating numerically the integral

$$G(y) = \frac{2}{\pi} \int_1^\infty dx \, \frac{\sqrt{x-1}}{x^2} \cos xy \, . \tag{13}$$

This impedance and its wake function, as well as the one described in the following subsection, are shown in Fig. 8.

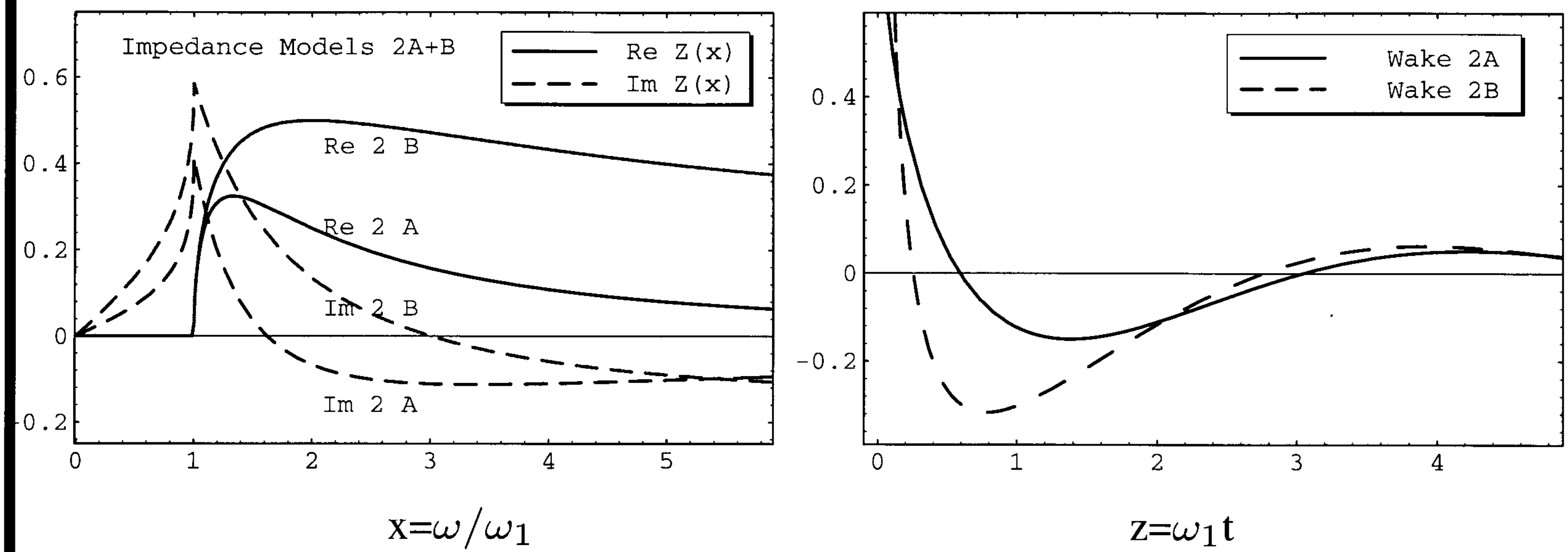

Figure 8: Normalised impedances and wake functions for models 2A+B.

Second BB Model for a Cavity

A similar model can be constructed for the impedance of a cavity, which tends asymptotically towards $\omega^{-1/2}$. For this case the impedance can be written:

$$\mathrm{Re}\, Z(x) = R_s \begin{cases} 0 & \text{for } |x| < 1\,, \\ \dfrac{\sqrt{|x|-1}}{|x|}\,, & \text{for } |x| > 1\,. \end{cases} \tag{14}$$

This function has a maximum at $x = 2$ and decreases as $x^{-1/2}$ at high frequencies. The expression for the imaginary part becomes:

$$\mathrm{Im} Z(x) = \frac{R_s}{x} \begin{cases} 2 - \sqrt{1+x} - \sqrt{1-x} & \text{for } |x| < 1\,, \\ 2 - \sqrt{1+x} & \text{for } |x| > 1\,. \end{cases} \tag{15}$$

The loss factor is obtained by integration of product of real part of impedance and charge distribution of bunch. For long Gaussian bunches this yields exponential decrease with bunch length, in agreement with numerical results.

Composite Model

An impedance model consisting of the sum of expressions for its various elements has been used to fit results of calculations and measurements of wakes in storage rings. This model has two attractive features: firstly, each term of the impedance has a clear physical meaning; secondly, the expressions for corresponding wake fields and loss factors can be derived analytically, which simplifies the fitting procedure to obtain the model parameters. However, the model diverges at low frequencies and needs to be supplemented by a cutoff.

The suggested expression for the longitudinal impedance is:

$$Z(\omega) = j\omega L + R_s + B\left[1 - j\,\mathrm{sign}(\omega)\right]\sqrt{|\omega|} + A\frac{1 + j\,\mathrm{sign}(\omega)}{\sqrt{|\omega|}}\,. \tag{16}$$

The model has 4 terms: the first one represents a low-frequency inductance, typical for shallow structures on accelerator vacuum chambers such as smooth tapers, bellows, vacuum ports, slots, collimators, etc. For small discontinuities the character remains inductive up to rather high frequencies.

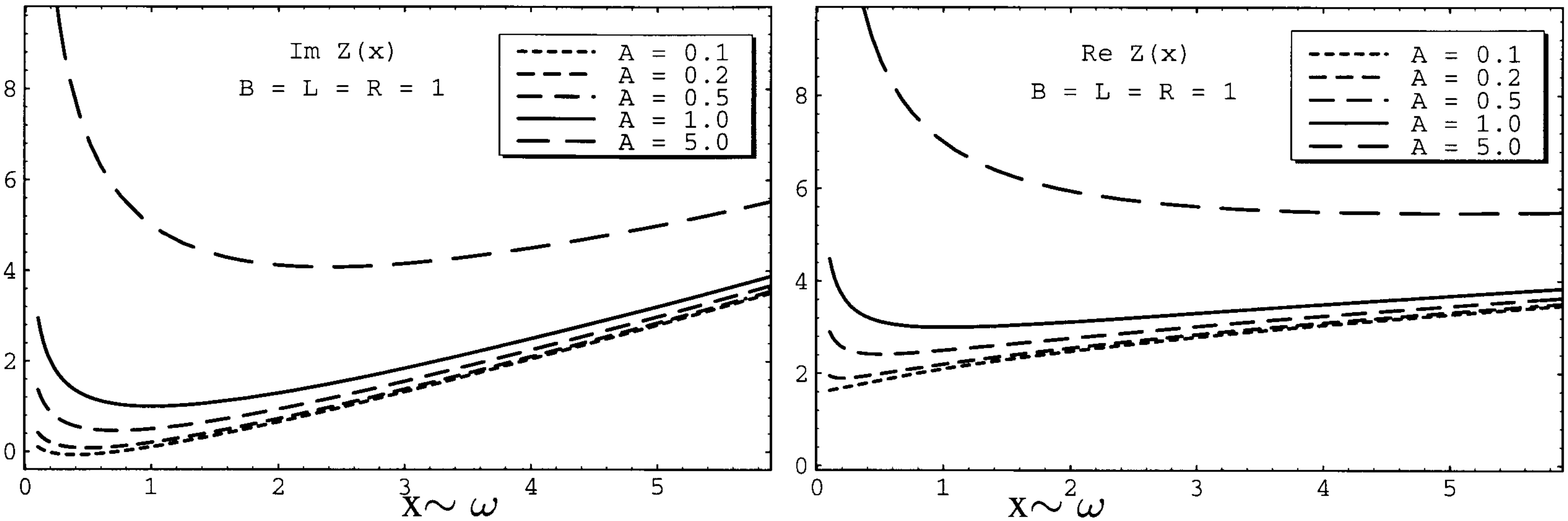

Figure 9: Real and imaginary parts of impedance for composite model.

The longitudinal wake potential of a Gaussian bunch with rms length $\sigma = \sigma_s/c$ is:

$$W^{ind}(\tau) = \frac{L\tau}{\sqrt{2\pi}\sigma^3} \exp\left(-\frac{\tau^2}{2\sigma^2}\right) . \tag{17}$$

It is an odd function of time delay τ, hence loss factor of Gaussian bunch is identically zero. The wake function has extrema at $\tau = \pm\sigma$ of magnitude $\hat{W}^{ind} = L/\sqrt{2\pi}\sigma^2$.

The second term in Eq. (16) is resistive. For a Gaussian it yields a wake potential:

$$W^R(\tau) = \frac{R_s}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\tau^2}{2\sigma^2}\right) . \tag{18}$$

Such a wake is produced by cavities for bunch lengths comparable to the beam pipe radius. The corresponding loss factor is:

$$\kappa_{\shortparallel}(\sigma) = \frac{R_s}{2\sqrt{\pi}\sigma} . \tag{19}$$

The third term in Eq. (16) describes impedance due to the finite wall resistivity. The corresponding wake potential can be expressed in terms of modified Bessel functions I_ν of fractional order ν and argument $\eta = (\tau/2\sigma)^2$:

$$W^{rw}(\tau) = B\frac{|\tau|^{3/2}}{4\sigma^3}\left[I_{-\frac{3}{4}}(\eta) - I_{\frac{1}{4}}(\eta)\, I_{-\frac{1}{4}}(\eta) + \mathrm{sign}(\tau) I_{\frac{3}{4}}(\eta)\right] \mathrm{e}^{-\eta} . \tag{20}$$

The loss factor of a Gaussian bunch of length σ becomes:

$$\kappa_{\shortparallel}(\sigma) = B\,\frac{\Gamma(3/4)}{2\pi\sigma^{3/2}} , \tag{21}$$

where $\Gamma(3/4) \approx 1.2254$.

The last term in Eq.(16) has the same high frequency dependence as the impedance of a single cavity. Its wake potential is given by:

$$W^{cav}(\tau) = A\frac{\sqrt{|\tau|}}{2\sigma}\left[I_{\frac{1}{4}}(\eta) + \mathrm{sign}(\tau)I_{-\frac{1}{4}}(\eta)\right]\mathrm{e}^{-\eta} \ , \tag{22}$$

and the corresponding loss factor becomes:

$$\kappa_{\shortparallel}(\sigma) = A\,\frac{\Gamma(1/4)}{2\pi\sigma^{3/2}} \ , \tag{23}$$

where $\Gamma(1/4) \approx 3.6256$.

The four parameters of the model A, B, L, R_s can be obtained by fitting the wake function $W(\tau)$ or the loss factor $\kappa_{\shortparallel}(\sigma)$ to values obtained either numerically or by measurements. The behaviour of this impedance is illustrated in Fig. 9 for several values of the parameter A, which is a coefficient of the cavity term, while all other parameters are set to unity.

Modified Inductance Model

Recently a model was designed to represent an inductive impedance at low frequencies, but with a non-vanishing loss factor there. The model has $\omega^{-1/2}$ dependence at high-frequencies. Its two free parameters L and T are determined by fitting to numerically computed data:

$$Z(\omega) = \frac{j\omega L}{(1+j\omega T)^{3/2}} \ . \tag{24}$$

The real and imaginary parts of this impedance, normalised by L/T are shown in Fig. 10. The corresponding wake function can be obtained explicitly by integration:

$$G(\tau) = \frac{L}{T} \begin{cases} 0 & \text{for } \tau < 0 \,, \\ \dfrac{\exp(-\tau/T)}{\sqrt{\pi\tau T}\,(1-2\tau/T)} & \text{for } \tau > 0 \,. \end{cases} \tag{25}$$

In the limit $T \to 0$ this function tends to the inductive wake $G(\tau) = L\delta'(\tau)$. The parameter L is the total ring inductance found by summing the contributions of all elements. The inductance of each of them can be estimated numerically with the help of various existing codes.

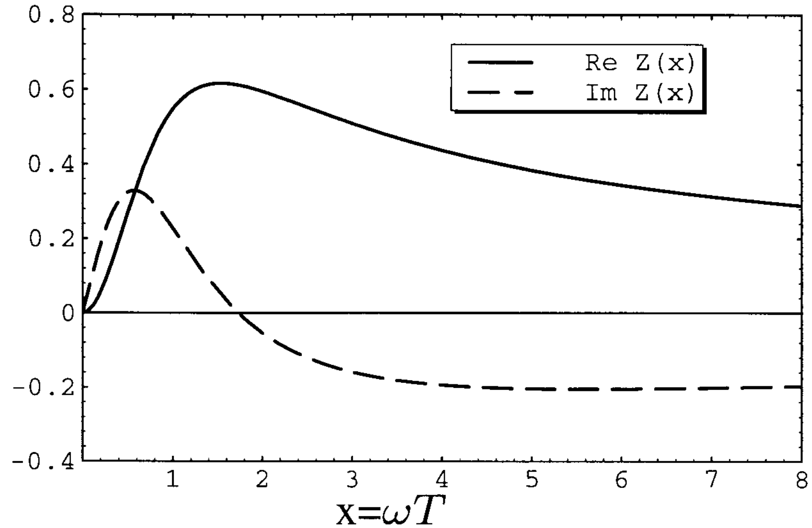

Figure 10: Normalised impedance of modified inductance model.

The second parameter, T, the inverse of cut-off frequency of vacuum chamber, can be found from loss factor. For long bunches, $T \ll \sigma$, it can be written:

$$\kappa_{\parallel}(\sigma) = \frac{3LT}{8\sqrt{\pi}\sigma^3} \ . \tag{26}$$

One can also find explicit expressions for corresponding transverse impedances and wake functions of this model.