On the coaxial wire measurement method of the longitudinal coupling impedance

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By means of a general theory we re-examine the coaxial wire method of measurement of the longitudinal coupling of a vacuum chamber components. We discuss the validity limits of the method in relation to the presence of the central wire that simulates the beam.

I. INTRODUCTION

The longitudinal coupling impedance (LCI) of an ultra-relativistic point charge q travelling along the z axis of a beam pipe is defined as [1]:

$$Z(\omega) = -\frac{1}{q}\int_{-\infty}^{+\infty} E_{sz}(r = 0, z, \omega) e^{jk_0 z} dz \tag{1}$$

where $\vec{E}_s$ is the Fourier transform of the electric field scattered by the discontinuities of the pipe and $k_0 = \omega/c$. The impedance of a generic component is often measured by means of the transmission S-parameter of a coaxial line obtained inserting a thin central wire. This method, which allows measuring $Z(\omega)$ with a simple bench measurements equipment, has been proposed in 70's on the ground of intuitive considerations. The basic idea is that the relativistic beam fields in the vacuum chamber can be simulated by means of a TEM wave propagating thanks to the presence of the wire [2,3,4]. Several formulae have been proposed to express the LCI as function of the above-mentioned transmission S-parameter. The first in order of time is the so-called relation of Sands and Rees suggested for the estimate of $Z(\omega)$ when the wire radius is very small [5]:

$$Z(\omega) = 2R_0\left(\frac{S_{2,1}^{REF} - S_{2,1}^{DUT}}{S_{2,1}^{REF}}\right) \tag{2}$$

$S_{2,1}^{DUT}$ being the transmission parameter of the component under test, $S_{2,1}^{REF}$ the parameter of a portion of unperturbed coaxial line of the same length, and $R_0 = \cancel{1} Z_0 ln\left(b/a\right)$ the characteristic impedance of the transmission line.

An improved expression valid for a single lumped impedance which is exact in the frame of the transmission line S-parameters, has been provided by Hahn and Pedersen [6]:

$$Z(\omega) = 2R_0\left(\frac{S_{2,1}^{REF} - S_{2,1}^{DUT}}{S_{2,1}^{DUT}}\right) \tag{3}$$

Other equations have been proposed for distributed impedances [7,8] for which the formula (3) breaks down.

Since the presence of the wire on the axis strongly affects the fields, it has been for long time discussed to which extent this technique is able to measure the impedance defined in eq.(1). A detailed analysis of the validity of the coaxial wire method, for azimuthally symmetric geometry, has been provided by Gluckstern and Li [9], who showed that the difference between (1) and (3) is of the order of $ln^{-1}\left(b/a\right)$.

Aim of this paper is to treat the problem from a more general point of wiev, confirming the validity of (3) independently of the shape of the test device. We use only the Schelkunoff's Field Equivalence Principle and the Lorentz Reciprocity Theorem.

II. LONGITUDINAL COUPLING IMPEDANCE BELOW CUTOFF

Let us consider an infinite beam pipe with a circular cross section of radius b and perfectly conducting walls. In the cylindrical system (r, φ, z) a point charge q moves with velocity $v = \pm c\hat{i}_z$ along z axis which coincides with the pipe axis. In the frequency domain, it has a spectrum given by:

$$\vec{J}_q^{\pm}(\vec{r},z;\omega) = \pm q\frac{\delta(r)}{2\pi r}e^{\mp jk_0 z}\hat{i}_z \tag{4}$$

and electromagnetic fields $\vec{E}_q^{\pm}$ and $\vec{H}_q^{\pm}$ on the pipe walls given by:

$$\vec{E}_q^{\pm}(r=b,\varphi,z;\omega) = Z_0\frac{q}{2\pi b}e^{\mp jk_0 z}\hat{i}_r \tag{5}$$

$$\vec{H}_q^{\pm}(r=b,\varphi,z;\omega) = \pm\frac{q}{2\pi b}e^{\mp jk_0 z}\hat{i}_\varphi \tag{6}$$

where Z_0 is the characteristic impedance of the vacuum.

Let us consider on the conducting wall an aperture S_A which can in general be coupled to an external structure. In order to account for the fields scattered by the aperture, we make use of the Schelkunoff's Field Equivalence Principle: a system of equivalent magnetic currents $\vec{J}_{ms}$ is placed on the aperture surface S_A and a perfectly conducting wall is introduced at an infinitesimal distance from these sources. These virtual currents are related to the electric field on the aperture $\vec{E}_A$ by means of the equation:

$$\vec{J}_{ms} = \hat{i}_r \times \vec{E}_A \qquad (7)$$

Then, the electromagnetic field into the waveguide can be written as:

$$\vec{E} = \vec{E}_q^+ + \vec{E}_s \qquad (8)$$

$$\vec{H} = \vec{H}_q^+ + \vec{H}_s \qquad (9)$$

where $\vec{E}_s$ and $\vec{H}_s$ are the fields scattered by the aperture, i.e. the fields radiated by the surface currents $\vec{J}_{ms}$ inside the pipe.

In order to calculate the LCI as function of the magnetic currents $\vec{J}_{ms}$ [10,11] we apply the Lorentz Reciprocity Theorem which relates the field $\vec{E}_1, \vec{H}_1$ generated by the electrical $\vec{J}_{e1}$ and magnetic $\vec{J}_{m1}$ currents, in a volume V bounded by a closed surface S, to the fields $\vec{E}_2, \vec{H}_2$ generated in the same volume by the sources $\vec{J}_{e2}$ and $\vec{J}_{m2}$:

$$\oint\!\!\oint_S \left(\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1\right) \cdot d\vec{S} = \iiint_V \left(\vec{H}_1 \cdot \vec{J}_{m2} - \vec{E}_1 \cdot \vec{J}_{e2} - \vec{H}_2 \cdot \vec{J}_{m1} + \vec{E}_2 \cdot \vec{J}_{e1}\right) dV \qquad (10)$$

or alternately:

$$\oint\!\!\oint_S \left(\vec{E}_1 \times \vec{H}_2^* + \vec{E}_2^* \times \vec{H}_1\right) \cdot d\vec{S} = -\iiint_V \left(\vec{E}_1 \cdot \vec{J}_{e2}^* + \vec{E}_2^* \cdot \vec{J}_{e1} + \vec{H}_1 \cdot \vec{J}_{m2}^* + \vec{H}_2^* \cdot \vec{J}_{m1}\right) dV \qquad (11)$$

In the relation (10) we choose the volume V (Fig.1) bounded by the perfectly conducting walls of the pipe, and the two cross-sections S_1 and S_2, on both sides of the aperture, and the following fields:

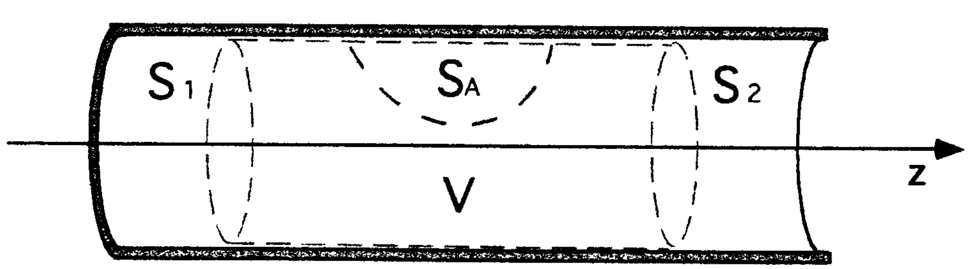

FIG.1. Volume V into the beam pipe.

1) $\vec{E}_q^-, \vec{H}_q^-$ generated by the source $\vec{J}_q^-$;

2) $\vec{E}_s, \vec{H}_s$ generated by the surface currents $\vec{J}_{ms}$..

Thus we obtain the following relation:

$$\iint_{S_1} \left(\vec{E}_q^- \times \vec{H}_s - \vec{E}_s \times \vec{H}_q^- \right) \cdot d\vec{S} + \iint_{S_2} \left(\vec{E}_q^- \times \vec{H}_s - \vec{E}_s \times \vec{H}_q^- \right) \cdot d\vec{S} \tag{12}$$
$$= \iint_{S_A} \vec{H}_q^- \cdot \vec{J}_{ms} dS - q \int_{z_1}^{z_2} E_{sz}(r=0,z) e^{jk_0 z} dz$$

Below the cut-off of the beam pipe, the fields $\vec{E}_s$ and $\vec{H}_s$ vanish exponentially for $z \to \pm\infty$. Therefore, moving the cross sections S_1 and S_2 to $-\infty$ and to $+\infty$ respectively, the integrals at the left-hand side of the equation (12) vanish, and we get:

$$-\frac{1}{q} \int_{-\infty}^{+\infty} E_{sz}(r=0,z) e^{jk_0 z} dz = -\frac{1}{q^2} \iint_{S_A} \vec{H}_q^- \cdot \vec{J}_{ms} dS \tag{13}$$

Recalling the definition (1) of the LCI, we can write:

$$Z(\omega) = \frac{1}{2\pi q b} \iint_{S_A} J_{ms\varphi} e^{jk_0 z} dS \tag{14}$$

We remark that the above equation is valid, below cut-off, for any shape of the aperture and of the external structure coupled through the aperture.

III. BEAM SIMULATED BY A WIRE

We assume, now, that in the same pipe a perfectly conducting wire of radius a is stretched along the z axis. The beam pipe so modified becomes a transmission line. In this new configuration, we consider a generator which excites a TEM wave $\vec{E}_q'^{+}, \vec{H}_q'^{+}$ such as to reproduce the fields $\vec{E}_q^{+}, \vec{H}_q^{+}$ of a point charge q in the unperturbed waveguide:

$$\vec{E}_q'^{+} = \frac{qZ_0}{2\pi r} e^{-jk_0 z} \hat{i}_r \tag{15}$$

$$\vec{H}_q'^{+} = \frac{q}{2\pi r} e^{-jk_0 z} \hat{i}_\varphi \tag{16}$$

Calling, now, $\vec{E}_A'$ the electric field on the aperture and $\vec{J}_{ms}'$ the equivalent magnetic surface currents, we have:

$$\vec{J}_{ms}' = \hat{i}_r \times \vec{E}_A' \tag{17}$$

The electromagnetic field inside the coaxial line can be expressed by the relations:

$$\vec{E}' = \vec{E}_q'^{+} + \vec{E}_s' \tag{18}$$

$$\vec{H}' = \vec{H}_q'^{+} + \vec{H}_s' \tag{19}$$

where $\vec{E}_s'$ and $\vec{E}_s'$ are the fields scattered by the aperture.

Below the cut-off frequency of the mode $TE_{1,1}$ of the coaxial line, the fields $\vec{E}_s'$ and $\vec{H}_s'$ at a sufficient distance from the aperture can be represented by means of the TEM components $\vec{E}_s'^{-}, \vec{H}_s'^{-}$ (backward) and $\vec{E}_s'^{+}, \vec{H}_s'^{+}$ (forward):

$$\vec{E}_s'^{\pm} = \frac{\alpha^{\pm}}{r} e^{\mp jk_0 z} \hat{i}_r \tag{20}$$

$$\vec{H}_s'^{\pm} = \pm \frac{\alpha^{\pm}}{Z_0 r} e^{\mp jk_0 z} \hat{i}_\varphi \tag{21}$$

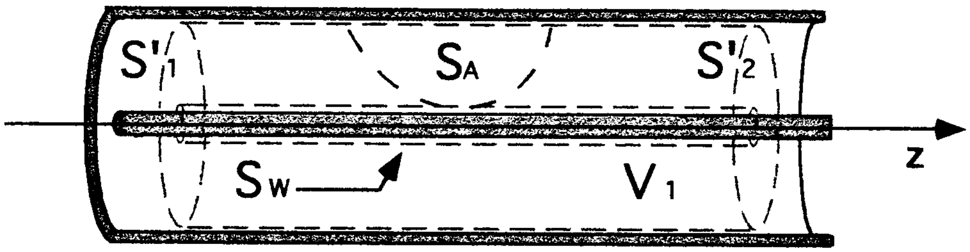

FIG. 2. Volume V_1 into the coaxial line.

The coefficients α^+ and α^- can be obtained by using again the Lorentz Reciprocity Theorem according to eq.(10). The volume V_1 in Fig.2 is similar to V of Fig.1, but for the surface S_W enclosing the wire. For α^+, considering the backward TEM wave $\vec{E}'^-, \vec{H}'^-$:

$$\vec{E}'^- = \frac{1}{r} e^{jk_0 z} \hat{i}_r \tag{22}$$

$$\vec{H}'^- = -\frac{1}{Z_0 r} e^{jk_0 z} \hat{i}_\varphi \tag{23}$$

and the field $\vec{E}'_s, \vec{H}'_s$ produced by the surface sources $\vec{J}'_{ms}$, we get the relation:

$$\iint_{S'_1} \left(\vec{E}'^- \times \vec{H}'^-_s - \vec{E}'^-_s \times \vec{H}'^- \right) \cdot d\vec{S} + \iint_{S'_2} \left(\vec{E}'^- \times \vec{H}'^+_s - \vec{E}'^+_s \times \vec{H}'^- \right) \cdot d\vec{S} = \iint_{S_A} \vec{H}'^- \cdot \vec{J}'_{ms} dS \tag{24}$$

By computing the integrals in the left-hand side we obtain:

$$\alpha^+ = -\frac{1}{4\pi b \ln\left(\frac{b}{a}\right)} \iint_{S_A} J'_{ms\varphi} e^{jk_0 z} dS \tag{25}$$

Analogously we get for α^-:

$$\alpha^- = \frac{1}{4\pi b \ln\left(\frac{b}{a}\right)} \iint_{S_A} J'_{ms\varphi} e^{-jk_0 z} dS \tag{26}$$

We observe that in case of vanishing wire radius, both α^+ and α^- tend to zero, because of the divergence of the logarithmic factor.

IV. IMPEDANCE FOR A SMALL BUT FINITE RADIUS OF THE WIRE

We apply twice the Reciprocity Theorem choosing the volumes V_1 (Fig.2), inside the pipe, and V_2 (Fig.3), outside the pipe, and assuming as sources of the fields the currents $(\vec{J}'_{ms}, \vec{J}_{ms})$ in V_1 and $(-\vec{J}'_{ms}, -\vec{J}_{ms})$ in V_2.

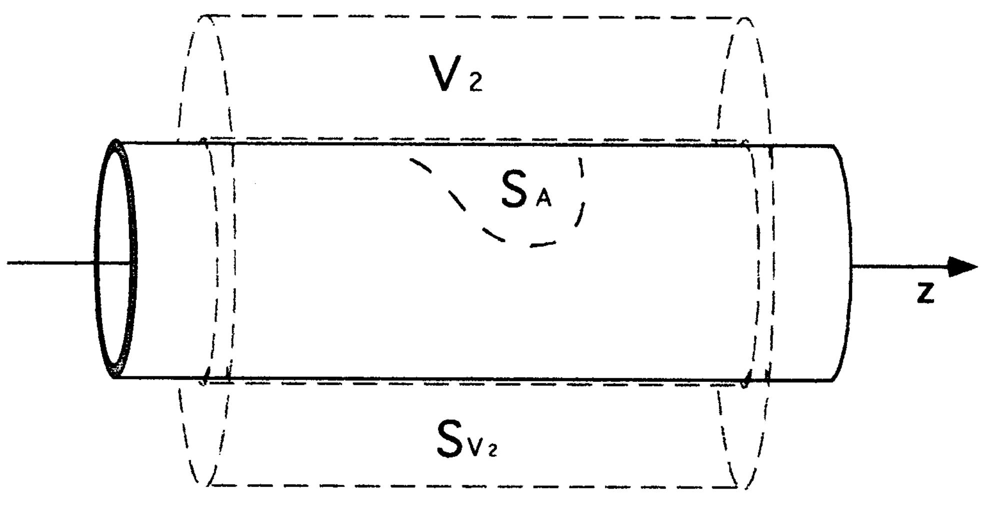

FIG.3. Volume V_2 outside the beam pipe.

We get for V_1:

$$\iint_{S_1'}\left(\vec{E}_s \times \vec{H}_s'^* + \vec{E}_s'^* \times \vec{H}_s\right)\cdot d\vec{S} + \iint_{S_2'}\left(\vec{E}_s \times \vec{H}_s'^* + \vec{E}_s'^* \times \vec{H}_s\right)\cdot d\vec{S} + \iint_{S_W} \vec{E}_s \times \vec{H}_s'^* \cdot d\vec{S} \tag{27}$$
$$= -\iint_{S_A} \vec{H}_s \cdot \vec{J}_{ms}'^* dS - \iint_{S_A} \vec{H}_s'^* \cdot \vec{J}_{ms} dS$$

and for V_2:

$$\iint_{S_{V_2}} \left(\vec{E} \times \vec{H}'^* + \vec{E}'^* \times \vec{H}\right) \cdot d\vec{S} = -\iint_{S_A} \vec{H} \cdot \vec{J}'^*_{ms} dS - \iint_{S_A} \vec{H}'^* \cdot \vec{J}_{ms} dS \tag{28}$$

S_{V_2} is the surface that encloses the volume V_2 wherein $-\vec{J}_{ms}$ and $-\vec{J}'_{ms}$ produce the electromagnetic fields $(\vec{E},\vec{H})$ and $(\vec{E}',\vec{H}')$; S'_1 and S'_2 are two cross sections of the coaxial line and S_W is the surface enfolding the wire.

Considering the magnetic fields $\vec{H}$ and $\vec{H}'$ given by eq.(9) and eq.(19), the relation (28) can be rewritten, dividing by $1/q^2$, as:

$$Z(\omega) + Z'^*(\omega) = \frac{1}{q^2}\left[\iint_{S_{V_2}} \left(\vec{E}'^* \times \vec{H} + \vec{E} \times \vec{H}'^*\right) \cdot d\vec{S} - \iint_{S_A} \vec{H}'^*_s \cdot \vec{J}_{ms} dS - \iint_{S_A} \vec{H}_s \cdot \vec{J}'^*_{ms} dS\right] \tag{29}$$

where

$$Z'(\omega) = \frac{1}{2\pi q b}\iint_{S_A} J'_{ms\varphi} e^{jk_0 z} dS \tag{30}$$

Furthermore, we observe that, under the following conditions:

1) thin wire,
2) wavelenght much greater than the longitudinal dimension of the aperture,
3) wavelenght much greater than the the cut-off wavelenght of the first higher-order mode excited by $\vec{J}_{ms}$,

the equation (27) becomes (Appendix A):

$$\frac{1}{q^2}\left[\iint_{S_A} \vec{H}'^*_s \cdot \vec{J}_{ms} dS + \iint_{S_A} \vec{H}_s \cdot \vec{J}'^*_{ms} dS\right] = -\frac{Z(\omega) Z'^*(\omega)}{2R_0} \tag{31}$$

and consequently the relation (29) changes into:

$$Z(\omega) + Z'^*(\omega) - \frac{Z'^*(\omega) Z(\omega)}{2R_0} = \frac{1}{q^2}\iint_{S_{V_2}} \left(\vec{E}'^* \times \vec{H} + \vec{E} \times \vec{H}'^*\right) \cdot d\vec{S} \tag{32}$$

Now, if the radius a of the central conductor is very small, the eigen-values and the eigen-functions of the coaxial line approach the circular waveguide ones (Appendix B) and, then, $\vec{J}'_{ms} \approx \vec{J}_{ms}$ and $(\vec{E}',\vec{H}') \approx (\vec{E},\vec{H})$. Replacing into the flux through S_{V_2} in eq.(32) $(\vec{E},\vec{H})$ with $(\vec{E}',\vec{H}')$ we obtain the following expression:

$$\iint_{S_{V_2}} \left(\vec{E}'^* \times \vec{H} + \vec{E} \times \vec{H}'^*\right) \cdot d\vec{S} \approx 2\,\mathrm{Re}\left[\iint_{S_{V_2}} \vec{E}' \times \vec{H}'^* \cdot d\vec{S}\right] \tag{33}$$

Using the Poynting Theorem in the volume V_2 as in the previous subsection with the current $-\vec{J}'_{ms}$ as source, we get:

$$2\,\mathrm{Re}\left[\iint_{S_{V_2}} \vec{E}' \times \vec{H}'^* \cdot d\vec{S}\right] = 2q^2\,\mathrm{Re}[Z'(\omega)] + 2\,\mathrm{Re}\left[\iint_{S_A} \vec{H}_s'^* \cdot \vec{J}'_{ms} dS\right] \tag{34}$$

On the other side, for the beam pipe with the wire along the axis, when the coupling apertures are much smaller than the wavelength:

$$\alpha^+ \approx -\alpha^- \tag{35}$$

and we get:

$$\mathrm{Re}\left[\iint_{S_A} \vec{H}_s'^* \cdot \vec{J}'_{ms} dS\right] = -\frac{8\pi^2 R_0 |\alpha^+|^2}{Z_0^2} \tag{36}$$

Therefore, the equation (34) becomes:

$$Z(\omega) + Z'^*(\omega) - \frac{Z'^*(\omega)Z(\omega)}{2R_0} = 2\,\mathrm{Re}[Z'(\omega)] - \frac{|Z'(\omega)|^2}{R_0} \tag{37}$$

Solving the above linear system for the unknowns $\mathrm{Re}[Z(\omega)]$ and $\mathrm{Im}[Z(\omega)]$ we get:

$$Z(\omega) = \frac{2R_0}{2R_0 - Z'^*(\omega)}\left[Z'(\omega) - \frac{|Z'(\omega)|^2}{R_0}\right] \tag{38}$$

which can be expressed by means of the transmission S-parameters as (Appendix C):

$$Z(\omega) = \frac{2R_0 S_{2,1}^{DUT} S_{2,1}^{REF^*}}{\left|S_{2,1}^{DUT}\right|^2}\left[\frac{S_{2,1}^{REF} - S_{2,1}^{DUT}}{S_{2,1}^{REF}} - 2\left|S_{2,1}^{REF} - S_{2,1}^{DUT}\right|^2\right] \tag{39}$$

Neglecting the term of second order into the square brackets, the above equation can be rewritten as:

$$Z(\omega) = 2R_0 \frac{S_{2,1}^{REF} - S_{2,1}^{DUT}}{S_{2,1}^{DUT}}\left(\frac{qZ_0/2\pi + \alpha^+}{qZ_0/2\pi + \alpha^{+*}}\right) \tag{40}$$

When $\left|\mathrm{Im}\left[\alpha^+\right]\right| << \left|qZ_0/2\pi + \mathrm{Re}\left[\alpha^+\right]\right|$, the relation (39) tends to the Hahn and Pedersen formula (3).

V. COMPARISON WITH THE EXPERIMENTAL DATA

We performed measurements with the coaxial wire method on a prototype of cylindrical beam pipe coupled by means of four slots to an external coaxial cavity [12]. The results are shown in Fig.4. We observe that the curve representing the eq.(39), obtained replacing $(\vec{E},\vec{H})$ with $(\vec{E}',\vec{H}')$, and the Hahn-Pedersen relation (3) are indistinguisheble.

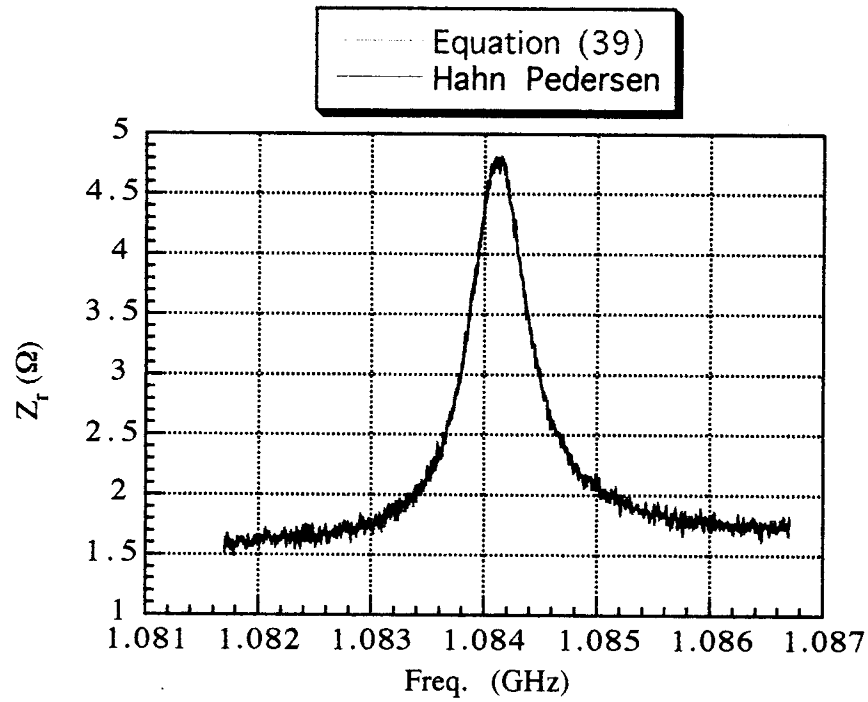

FIG.4. Real part of the impedance calculated using the Hahn and Pedersen formula and the eq.(39).

VI. CONCLUSIONS

In this paper we have given a theoretical proof of the coaxial wire method for lumped impedances, used in the bench measurements of the coupling impedance, in the limit of low frequency but for any shape of the test device. Using the Equivalence Principle and the Reciprocity Theorem, we have derived an approximate expression which take into account the presence of the wire. For a real measurement set-up, with a thin coaxial wire, this one tends to the well-known Hahn and Pedersen formula derived in the frame of the S-parameters of a transmission line.

VII. ACKNOWLEDGMENT

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APPENDIX A

Let us consider the equation (33). Moving S_1' and S_2' to $-\infty$ and $+\infty$ respectively, and exploiting the vectorial property:

$$\vec{E}_s \times \vec{H}_s'^* \cdot d\vec{S} = -\vec{E}_s \times \vec{H}_s'^* \cdot \hat{i}_r dS = \hat{i}_r \times \vec{H}_s'^* \cdot \vec{E}_s dS \tag{A1}$$

we obtain the relation:

$$\iint_{S_A} \vec{H}_s'^* \cdot \vec{J}_{ms} dS + \iint_{S_A} \vec{H}_s \cdot \vec{J}_{ms}'^* dS = -\iint_{S_W} \hat{i}_r \times \vec{H}_s'^* \cdot \vec{E}_s dS \tag{A2}$$

where S_W is the whole wire surface.

We assume that when the wire radius a is very small:

$$\vec{E}_s(r = a, \varphi, z) \approx \vec{E}_s(r = 0, z) \tag{A3}$$

and neglecting the contributions of the evanescent modes and recalling the equation (21), we can express the vector product $\hat{i}_r \times \vec{H}_s'^*$ in (A2) on the wire surface as:

$$\left[-\frac{\alpha^{-*}}{Z_0 a} e^{-jk_0 z} u(-z) + \frac{\alpha^{+*}}{Z_0 a} e^{jk_0 z} u(z) \right] \hat{i}_z \tag{A4}$$

where $u(z)$ is the step function.

We observe, at last, that, if the longitudinal dimension of the aperture, centered, for convenience, on $z=0$, is very small compared with the wavelength, the equation (46) becomes true and the right-hand side of the equation (A1), divided by $1/q^2$, may be rewritten as follows:

$$-\frac{2\pi\alpha^{+*}}{Z_0 q^2}\left[\int_{-\infty}^{0} E_{sz}(r=0,z)e^{-jk_0 z}dz+\int_{0}^{+\infty} E_{sz}(r=0,z)e^{jk_0 z}dz\right] \tag{A5}$$

Now, adding and subtracting the quantity

$$\int_{-\infty}^{0} E_{sz}(r=0,z)e^{jk_0 z}dz \tag{A6}$$

inside the square brackets, recalling the expressions (1) and (27), and taking into account that the field E_{sz} is an even function of z [13], the (A5) becomes:

$$-\frac{Z(\omega)Z'^{*}(\omega)}{2R_0}+j\frac{2\pi\alpha^{+*}}{Z_0 q^2}\int_{-\infty}^{+\infty} E_{sz}(r=0,z)\sin(k_0|z|)dz \tag{A7}$$

Now we note that $E_{sz}(r=0,z)$ is significantly different from zero only in a narrow interval around $z=0$, being the electric scattered field $\vec{E}_s$ evanescent below the pipe cutoff. Then, if the wavelength is much longer than the cutoff wavelength of the first higher-order mode excited by the surface sources $\vec{J}_{ms}$, then, the function $\sin(k_0|z|)$ may be considered nearly zero in the interval above-mentioned and the integral between $+\infty$ and $-\infty$ negligible. Hence the equation (A2) becomes:

$$\frac{1}{q^2}\left[\iint_{S_A}\vec{H}_s'^{*}\cdot\vec{J}_{ms}dS+\iint_{S_A}\vec{H}_s\cdot\vec{J}_{ms}'^{*}dS\right]=-\frac{Z(\omega)Z'^{*}(\omega)}{2R_0} \tag{A8}$$

APPENDIX B

Let us consider a coaxial line of outer radius b and inner radius a with perfectly conducting walls. We want to study the behaviour of the higher-order modes when $a \to 0$.

In order to do that, we solve, on the transverse plane, separating the variables, the two Helmholtz's equation for TE and for TM modes respectively. The solutions can be expressed as linear combination of Bessel's functions of first and second kind:

$$B_1 J_n(k_t' r) + B_2 Y_n(k_t' r) \tag{B1}$$

where B_1 and B_2 are two constants and k_t' are the eigenvalues of the problem.

Then, applying the boundary conditions, we get, in the case of TE modes, the characteristic equation:

$$J_n'(ak_t')Y_n'(bk_t') - J_n'(bk_t')Y_n'(ak_t') = 0 \tag{B2}$$

Dividing by $Y_n'(bk_t')Y_n'(ak_t')$, we obtain:

$$\frac{J_n'(bk_t')}{Y_n'(bk_t')} = \frac{J_n'(ak_t')}{Y_n'(ak_t')} \tag{B3}$$

For vanishing radius a the RHS ratio tends to zero. In fact, while $J_n'(ak_t')$ converges to a finite value, $Y_n'(ak_t')$ diverges. Then, the equation (B3) becomes:

$$J_n'(bk_t) = 0 \tag{B4}$$

This is the characteristic equation of the circular waveguide, whose the eigenvalues k_t' tend to the corresponding k_t of the structure simply connected.

Furthermore, it's possible to prove that:

$$\frac{J_n'(ak_t')}{Y_n'(ak_t')} = -\frac{B_2}{B_1} \tag{B5}$$

Hence, we have that, as $a \to 0$, the coefficient of the Bessel's function of second kind in eq. (B1) disappears. In conclusion, we can say that the modes TE of the coaxial line tend to the homologous modes of the circular waveguide.

Now, proceeding in a similar manner, for the TM modes we obtain the following characteristic equation:

$$J_n(ak_t')Y_n(bk_t') - J_n(bk_t')Y_n(ak_t') = 0 \tag{B6}$$

and, then, the relation:

$$\frac{J_n(bk_t')}{Y_n(bk_t')} = \frac{J_n(ak_t')}{Y_n(ak_t')} = -\frac{B_2}{B_1} \tag{B7}$$

Since $J_n(ak_t')$ tends to a finite value, when a approach zero, and $Y_n(ak_t')$ diverges, we have that the eigenvalues and the eigenfunctions of the coaxial line converge to the homologous quantities of the circular waveguide.

At this point, we want to focus our attention to the convergence of the coaxial line $TM_{0,m}$ eigenfunctions to the homologous solutions of the waveguide. In order to do that, we fix the outer radius b equal to 0,05 m and we consider the following different values for the inner radius a.:

$$a_1 = 10^{-3} m \qquad a_2 = 10^{-4} m \qquad a_3 = 10^{-7} m \qquad a_4 = 10^{-20} m$$

Afterwards, we'll limit our analysis to the $TM_{0,1}$ mode, supposing that its behavior should be representative of all the $TM_{0,m}$.

Then, we can calculate, solving the equation (B5), the eigenvalues $k_{t0,1}'$ of the coaxial line in relation to the different radius a and the eigenvalue $k_{t0,1}$ of the circular waveguide whit radius b = 0.05 m. The values founded, reported in Fig.5, satisfy the property of convergence above discussed.

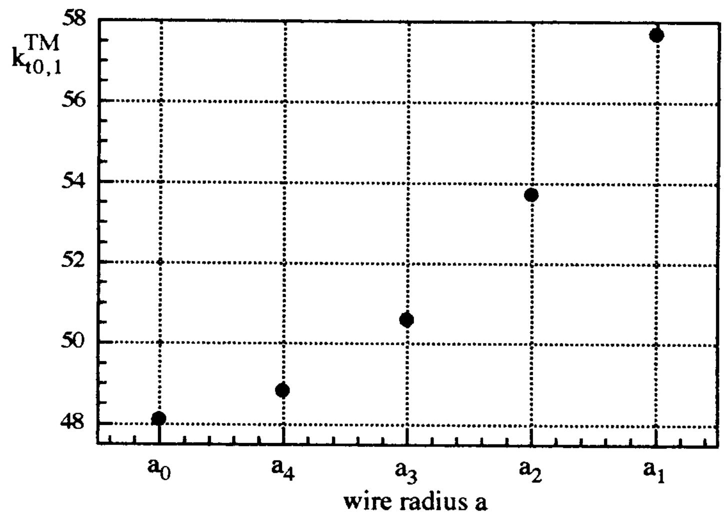

FIG.5. Eigenvalue of the $TM_{0,1}$ mode for the wire radius a. Corresponding to a_0 we have indicated the eigenvalue of the cylindrical waveguide.

Now, we compare the radial behavior of the longitudinal electric field in coaxial line with the same field component in the circular waveguide. In Fig.6 we have represented the function

$$J_0\left(k'_{t0,1}r\right)-\frac{J_0\left(k'_{t0,1}b\right)}{Y_0\left(k'_{t0,1}b\right)}Y_0\left(k'_{t0,1}r\right) \tag{B8}$$

for the different value of the wire radius and the function

$$J_0\left(k_{t0,1}r\right) \tag{B9}$$

for the simply connected pipe. We observe that, when $a \le 10^{-4}m$, the field of the coaxial line, close to the external walls of the pipe, approaches the field of the circular waveguide while the electromagnetic field is modified only in a narrow region around the wire.

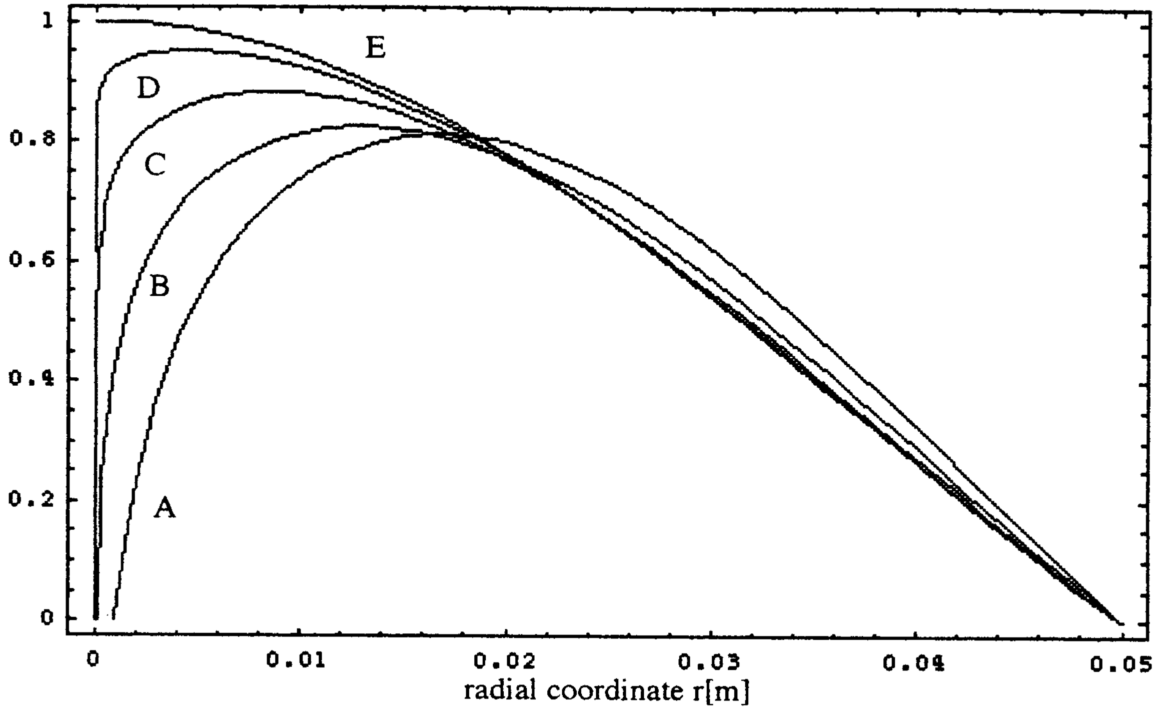

FIG.6. Radial behaviour of the $TM_{0,1}$ mode longitudinal electric field e_z for different value of the wire radius:

A- $a=a_1$ B- $a=a_2$ C- $a=a_3$ D- $a=a_4$

The curve E represent the field e_z in the cylindrical waveguide.

APPENDIX C

Comparing (30) with (25) it's readily found that:

$$Z'(\omega) = -2\frac{R_0\alpha^+}{qZ_0/2\pi} \tag{C1}$$

Moreover, since the coaxial structure of Fig.2 is as a two-port network, it's possible to express the right-hand side of the equation (C1) by means of the circuital S-parameters. In fact, if the output line is terminated on a matched load and we use a matched generator producing the same fields (15) and (16), the transmission parameter of the Scattering Matrix* is given by:

* Notice the different meaning of + and - for the scattering parameters, the sign + means incident waves on the device ports.

$$S_{2,1}^{DUT} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^+=0} = \frac{\left(qZ_0/2\pi + \alpha^+\right)}{qZ_0 e^{-jk_0 z_1'}/2\pi} e^{-jk_0 z_2'} = \left(1 + \frac{\alpha^+}{qZ_0/2\pi}\right) e^{-jk_0 l} \qquad \text{(C2)}$$

where $l = z_2' - z_1'$.

We remind that for a coaxial line of length l, the transmission parameter is given by:

$$S_{2,1}^{REF} = e^{-jk_0 l} \qquad \text{(C3)}$$

thus, from (C1), (C2) and (C3) we obtain:

$$Z'(\omega) = 2R_0\left(\frac{S_{2,1}^{REF} - S_{2,1}^{DUT}}{S_{2,1}^{REF}}\right) \qquad \text{(C4)}$$

and readily eq.(39) from eq.(38).

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