Wake Fields Effects due to Surface Roughness

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The problem of the wake field generated by a relativistic particle travelling in a long beam tube with rough surface has been revisited by means of a standard theory based on the hybrid modes excited in a periodically corrugated waveguide. Slow waves synchronous with the particle can be excited in the structure, producing wake fields whose frequency and amplitude depend on the depth of the corrugation.

1 Historical review

- **K.L.F.Bane, C.K.Ng, and A.W.Chao,** " Estimate Impedance Due to Wall Surface Roughness", SLAC-PUB-7514 (1997) and PAC'97, Vancouver, Canada.

  In machine with short bunches, the surface roughness may be a major source of wakefields which might significantly increase the beam emittance and the energy spread. They estimated the low frequency Impedance due to wall surface roughness.

- **KA. Novokhatski and A. Mosnier,** "Wakefields of Short Bunches in the Canal Covered with thin Dielectric Layer", PAC'97, Vancouver, Canada.

- **KA. Novokhatski, M. Timm and T. Weiland,** "The Surface Roughness Wake Field Effect", Proceedings of the IACP'98

  The monopole wake fields of a waveguide with a rough surface behave in the same way as those inside a waveguide with a dielectric layer. Through time-domain simulations, they find a longitudinal wake function:

$$\frac{\partial w(\tau)}{\partial z} = \frac{Z_0 c}{\pi a^2} \cos\left(2\pi \overline{f_1} \tau\right) H(\tau) \tag{1}$$

  which might be could be cause of emittance degradation in the TESLA-FEL, and in LCLS.

In our work we review the problem of the wakefields produced by an ultrarelativistic charge travelling inside a beam tube with a periodic corrugation making use of a standard theory based on the Hybrid modes propagating in the waveguide. The work has been develped with particular regard to LHC beam pipe where, a corrugations has been proposed in order to reduces the walls reflectivity.

A similar theory has been already used in the design of a wide band pick-up for sthocastic cooling.

2 The Relevant Geometry

We assume $t \ll L$ and neglect ohmic losses in the material.

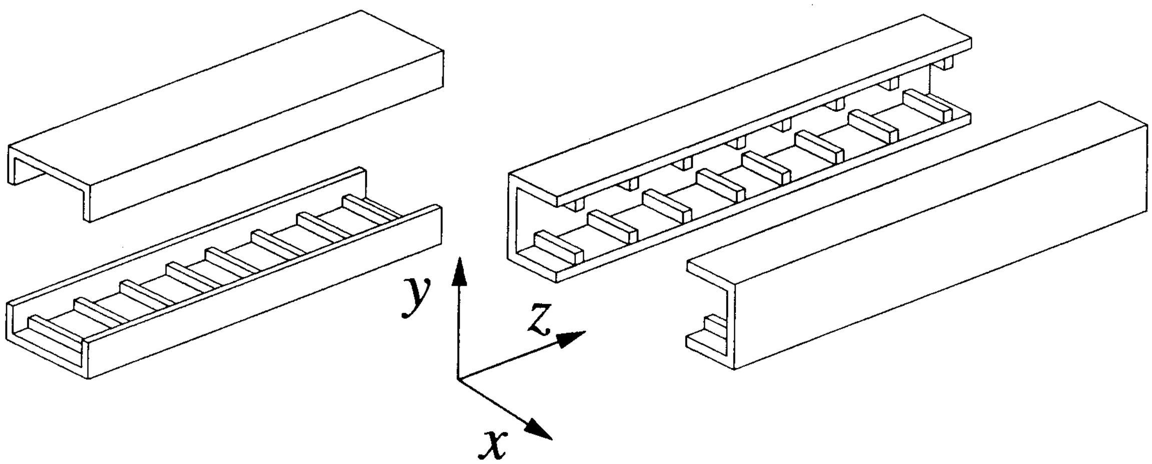

Figure 1: Relevant geometry.

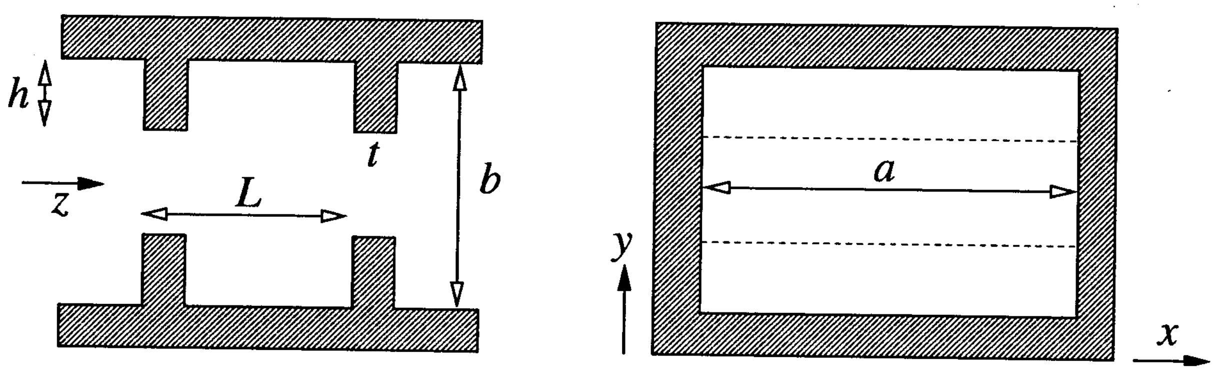

Figure 2: Schematic view of the waveguide and notations adopted.

3 Homogeneous problem

Due to the rectangular geometry, the fields inside a single corrugation can be written as

$$\begin{aligned}
E_z^c &= -j\omega\mu_0 \sum_n B_n \sin\left[\beta_n \left(b/2 + h - y\right)\right] \cos\left(k_{xn} x\right) && \text{(2a)}\\
E_y^c &= 0 && \text{(2b)}\\
H_z^c &= 0 && \text{(2c)}\\
H_y^c &= \frac{\pi}{a} \sum_n n B_n \sin\left[\beta_n \left(b/2 + h - y\right)\right] \sin\left(k_{xn} x\right) && \text{(2d)}\\
H_x^c &= -\beta_n \sum_n B_n \cos\left[\beta_n \left(b/2 + h - y\right)\right] \cos\left(k_{xn} x\right) && \text{(2e)}
\end{aligned}$$

where

$$\beta_n = \sqrt{\left(\omega/c\right)^2 - k_{xn}^2} \quad \text{and} \quad k_{xn} = n\pi/a \tag{3}$$

The fields of interest in the internal region of the waveguide can be derived from the only magnetic Hertz potential $\mathbf{\Pi}_m = \hat{\boldsymbol{x}}\Pi_x$:

$$\begin{aligned}
\boldsymbol{E} &= -j\omega\mu\nabla \times \mathbf{\Pi}, && \text{(4a)}\\
\boldsymbol{H} &= (k^2 + \nabla\nabla\cdot)\mathbf{\Pi}. && \text{(4b)}
\end{aligned}$$

$$\Pi_x = \left[A\cos\left(k_x x\right) + B\sin\left(k_x x\right)\right]\left[C\cos\left(\alpha y\right) + D\sin\left(\alpha y\right)\right] e^{-j\beta_z z}, \tag{5}$$

where A, B, k_x, β_z and α are constants. The separability condition for the wave number inside the waveguide can be written as:

$$\beta_z^2 + \alpha^2 + k_{xn}^2 = \left(\omega/c\right)^2. \tag{6}$$

Applying the continuity of fields over the same boundary,i.e.

$$E_z^c = E_z \quad \text{and} \quad H_x^c = H_x \quad \text{for} \quad y = \pm b/2 \tag{7}$$

we get

$$\beta_n \tan(\beta_n h) = \alpha \cot(\alpha b/2)\,. \tag{8}$$

usually referred as dispersion equation.

To show the basic features of the solution, we study the case of square waveguide and we plot the normalised propagation constant $\beta_z a$ as a function of a normalised frequency a/λ for the case $h/b = 0.1$

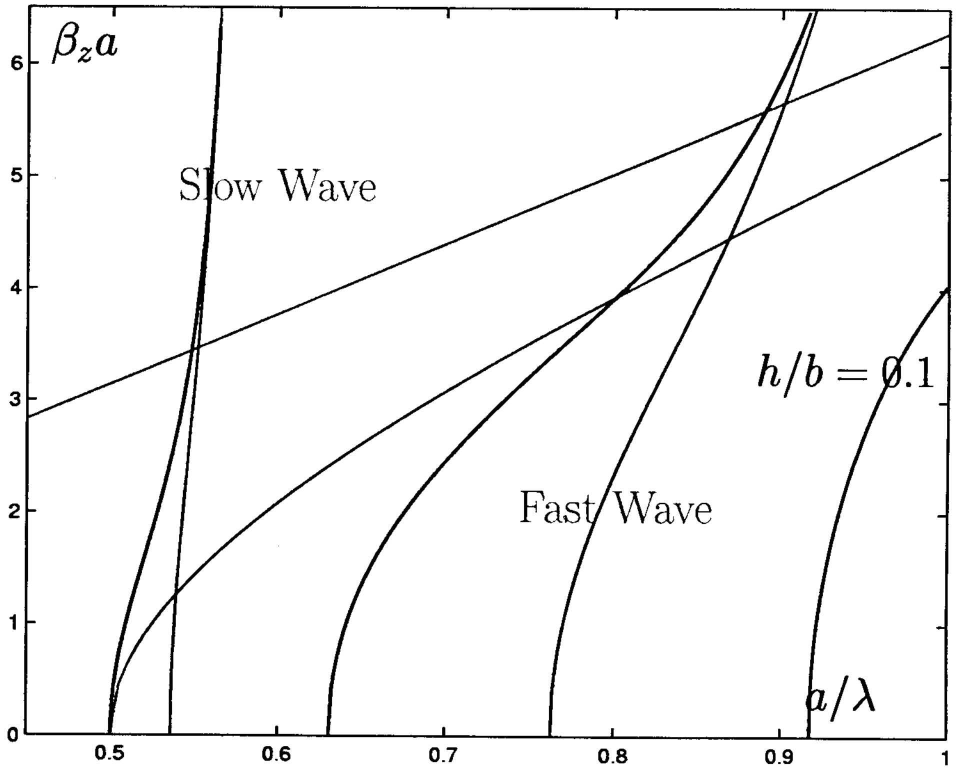

Figure 3: (Color) Brilluion diagram for a square waveguide of side a with corrugation on two opposite faces of depth h. The green curve is the propagation constant of the fundamental mode of the corresponding smooth waveguide and the red line is the dispersion line of a relativistic beam.

It can be shown that for $h \ll \lambda, a$ and high energy particles ($\gamma \to \infty$) the crossing frequencies are:

$$\overline{f_n} = \frac{c}{2\pi}\sqrt{k_{xn}^2 + \frac{k_{xn}}{h}\left[\tanh\left(k_{xn}\frac{b}{2}\right)\right]^{-1}}\,. \tag{9}$$

For very small h, the second term in the square root dominates resulting in the typical behaviour $\propto 1/\sqrt{h}$.

The minimum frequency beyond which a mode can propagate in the guide (namely its cut-off frequency f_c) depends as well on the corrugation:

$$f_c = \frac{c}{2\pi}\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{2h+b}\right)^2} \quad \text{with} \quad n = 0, 1, 2, \ldots \quad \text{and} \quad m = 0, 1, 2, \ldots; \tag{10}$$

mathematically they are found imposing the condition $\beta_z = 0$. In the limit of $h \to 0$, we find the well-known cut-off frequencies for modes of a smooth waveguide. Only the waveguide cut-off with ($m = 0$) does not depends on h.

For small h/b we can write a formula of β_z as a function of the propagation constant of the smooth waveguide, namely k_z:

$$\beta_z \approx k_z + \frac{2\beta_n}{k_z b}\tan\left(\beta_n h\right). \tag{11}$$

If the propagation constant of our hybrid mode falls in the region below (above) the red line, the wave is usually referred as fast (slow) because its phase velocity is bigger (smaller) than the light one. Moreover, β_z for a rough waveguide can exceed the wave number of the smooth one, meaning that the solution α is purely imaginary. In that case the wave is told to be a surface wave, because it is exponentially dumped in the y-direction. Thus, only a surface wave can be also a slow one.

4 Including the sources

Once solved the homogeneous problem, the field generated by a point charge can be found by means of the Lorentz reciprocity principle:

$$\oint_S \left(\boldsymbol{E}_n^{\pm} \times \boldsymbol{H} - \boldsymbol{E} \times \boldsymbol{H}_n^{\pm}\right) \cdot \boldsymbol{n}\mathrm{d}S \;=\; \int_V \boldsymbol{J} \cdot \boldsymbol{E}_n^{\pm}\mathrm{d}V \tag{12}$$

where $\boldsymbol{J}$ is the current density of a point charge travelling on axis ($\hat{z}_o$ unit vector along z-axis),

$$\boldsymbol{J}(x, y, z; \omega) = q\,\delta(x)\delta(y)e^{-j\frac{\omega}{\beta c}z}\hat{z}_o; \tag{13}$$

In a rough waveguide the (hybrid) modes are nomore orthogonal, however the coupling itself goes to zero in the limit $h \to 0$.

Moreover, only the modes synchronous with the beam can exchange energy over an infinite interaction length (*surfing effect*). The electric field shows a resonant behaviour around the crossing synchronous frequency:

The synchronism between the field in the waveguide and the beam is possible because of the slowing effect due to the surface roughness.

For relativistic particles ($\gamma \to \infty$) and small h ($h \ll \lambda$) the electric field at the lowest frequency ($n = 1$) is

$$E_z\left(x, y, z; \omega\right) \sim -4\pi^2\, q\, Z_0\, \frac{h}{a}\, \frac{1}{ab}\, \tanh\left(\frac{\pi}{2}\frac{b}{a}\right) \left[\frac{\sinh\left(\pi b/a\right)}{\pi b/a} - 1\right]^{-1} \times$$
$$\cos\left(\frac{\pi}{a}x\right)\, \cosh\left(\frac{\pi}{b}y\right)\, e^{-jz\omega/c}\left[\delta\left(\omega/c \;-\; \beta_{z1}\right) + \delta\left(\omega/c \;+\; \beta_{z1}\right)\right], \tag{14}$$

E_z has a π phase difference with the charge (the - sign), meaning that it is a decelerating field. The field is confinated in the waveguide region near the corrugated wall (it is exsponentially growing for $y \to \pm b/2$), as we expect for a surface wave. Moreover it depends on the geometry of the pipe (ab is the area the pipe cross section). The height of the corrugation fixes not only the resonant frequency through β_{z1}, but also the field amplitude through the factor h/a.

5 Longitudinal coupling impedance and wake function

The longitudinal coupling impedance per unit length is defined by:

$$\frac{\partial Z(\omega)}{\partial z} = -\frac{1}{q} E_z \left(x=0, y=0, z, \omega\right) e^{jz\omega/c}; \tag{15}$$

we get

$$\begin{aligned}\frac{\partial Z(\omega)}{\partial z} =& 4\pi^2\ Z_0\ \frac{h}{a}\ \frac{1}{ab}\ \tanh\left(\frac{\pi}{2}\frac{b}{a}\right)\left[\frac{\sinh\left(\pi b/a\right)}{\pi b/a} - 1\right]^{-1} \times \\ & \left[\delta\left(\omega/c\ -\ \beta_{z1}\right) + \delta\left(\omega/c\ +\ \beta_{z1}\right)\right].\end{aligned} \tag{16}$$

which is purely real. It is straightforward now to get the wake function for unit length :

$$\frac{\partial w(\tau)}{\partial z} = \frac{H(\tau)}{\pi} \int_{-\infty}^{infty} \frac{\partial Z(\omega)}{\partial z} e^{j\omega\tau} \mathrm{d}\omega \tag{17}$$

where τ is the time distance of the trailing charge from the leading one and $H(\tau)$ is the Heaviside function. We get

$$\begin{aligned}\frac{\partial w(\tau)}{\partial z} &= 8\pi \frac{Z_0\ c}{ab}\ \frac{h}{a}\ \tanh\left(\frac{\pi}{2}\frac{b}{a}\right)\left[\frac{\sinh\left(\pi b/a\right)}{\pi b/a} - 1\right]^{-1}\ \cos\left(2\pi\overline{f_1}\,\tau\right)\ H(\tau) = \\ &= w_0\left(a,b,h\right)\ \cos\left(2\pi\overline{f_1}\,\tau\right)\ H(\tau).\end{aligned} \tag{18}$$

The amplitude of the sinusoidal function for an LHC-like geometry ($a = 3.6\ 10^{-2}$ m, $b = 2.5\ 10^{-2}$ m and $h = 30\ \mu$m) is $\simeq 2\ V\ pC^{-1}\ m^{-1}$.

6 Transverse coupling impedance and wake function

To solve the transverse problem, differentely from the longitudinal situation, we consider the point charge moving off-axis, with a displacement y_0. Due to the new simmetry, we choose the first magnetic Hertz potential:

$$\Pi_{mx1} = \sum_n A_n \cos\left(\frac{n\pi}{a}x\right)\cos\left(\xi_{yn}y\right)e^{-j\xi_{znm}z} \qquad n = 1, 3, \ldots \tag{19}$$

From Eq. (4), we derive the components of the electromagnetic field.

The boundary conditions at the walls lead to the following dispersion equation:

$$\beta_n \tan(\beta_n h) = -\alpha \tan(\alpha \frac{b}{2}) \tag{20}$$

For $h \ll b$, the lowest dipole mode is the TE_{10}, with a cut-off frequence given by:

$$f_c = \frac{c}{2\pi}\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{2h+b}\right)^2} \quad \text{with} \quad n = 1 \quad \text{and} \quad m = 0; \tag{21}$$

i.e.:

$$f_{c10} = \frac{c}{2a}. \tag{22}$$

And the crossing frequence (where $k = k_z$), at:

$$\overline{f_n} = \frac{c}{2\pi}\sqrt{k_{xn}^2 + \frac{k_{xn}}{h}\left[\tanh\left(k_{xn}\frac{b}{2}\right)\right]}. \tag{23}$$

Analogously to the longitudinal case, for very small h, the second term in the square root dominates, and we find the typical behaviour $\propto 1/\sqrt{h}$.

Applying the Lorentz reciprocity principle (see Eq. (12)), the field generated by a point charge can be found. Now $\boldsymbol{J}$ is the current density of a point charge travelling not along the z-axis:

$$\boldsymbol{J}(x,y,z;\omega) = q\,\delta(x)\delta(y-y_0)e^{-j\frac{\omega}{\beta c}z}\hat{z}_o; \tag{24}$$

Approximating for relativistic particles and for small h and y_0 at the lowest frequency (n=1), we find the transverse impedence per unit length, along the y-axis:

$$\frac{\partial Z_y(\omega)}{\partial z} = -4\pi^{\frac{5}{2}} Z_0 \frac{h^{\frac{3}{2}} y_0}{a^{\frac{7}{2}} b} \left\{ \left[\tanh\left(\frac{\pi b}{2a}\right)\right]^{\frac{3}{2}} \left[\frac{\sinh\left(\frac{\pi b}{a}\right)}{\frac{\pi b}{b}}\right] \right\}^{-1} \times$$
$$\times \left[\delta\left(\frac{\omega}{\beta c} - \xi_{zn}\right) - \delta\left(\frac{\omega}{\beta c} + \xi_{zn}\right)\right] \tag{25}$$

And the transverse wake function per unit length along the y-axis:

$$\frac{\partial w_y(\tau)}{\partial z} = -4\pi^{\frac{5}{2}} Z_0 c \frac{h^{\frac{3}{2}} y_0}{a^{\frac{7}{2}} b} \left\{ \left[\tanh\left(\frac{\pi b}{2a}\right)\right]^{\frac{3}{2}} \left[\frac{\sinh\left(\frac{\pi b}{a}\right)}{\frac{\pi b}{a}}\right] \right\}^{-1} \sin(\omega_0 \tau) \tag{26}$$

Analogously to the rectangular waveguide, the problem in the circular one can be studied using the Hertz potentials. The Hertz potential chosen is the electrical potential along the z-axis:

$$\Pi_{ez} = \sum_n A_n J_n(k_t r) cos(n\phi) e^{j(\omega t - \beta'_n z)} \tag{28}$$

where β'_n is the hybrid mode propagation constant in the circular waveguide and k_t is the transverse propagation constant.
The field components are found by Eq. (4):

$$e_{rn} = -j\beta'_n k_t J'_n(k_t r)\cos(n\phi) \tag{29a}$$

$$e_{\phi n} = j\beta'_n \frac{n}{r} J_n(k_t r)\sin(n\phi) \tag{29b}$$

$$e_{zn} = k_t^2 J_n(k_t r)\cos(n\phi) \tag{29c}$$

$$h_{rn} = -j\omega\epsilon \frac{n}{r} J_n(k_t r)\sin(n\phi) \tag{29d}$$

$$h_{\phi n} = -j\omega\epsilon k_t J'_n(k_t r)\cos(n\phi) \tag{29e}$$

$$h_{zn} = 0 \tag{29f}$$

it is evident that it is a TM mode.
Applying the boundary conditions , the dispersion reletion is found to be:

$$\frac{J_n(k_0 a)Y_n(k_0 b) - J_n(k_0 b)Y_n(k_0 a)}{J'_n(k_0 a)Y_n(k_0 b) - J_n(k_0 b)Y'_n(k_0 a)} = \frac{k_t}{k_0}\frac{J_n(k_0 a)}{J'_n(k_t a)} \tag{30}$$

where a is the inner radius and b the outer radius.

From Eq. (30), for small h and n=0, the crossing frequence for the circular waveguide can be calculated:

$$\omega_0 = c\sqrt{\frac{2}{ah}} \tag{31}$$

The field generated by a point charge can be found applying the Lorentz reciprocity principle Eq. (12), where $\boldsymbol{J}$ is the current density of a point charge travelling on axis ($\hat{z}_o$ unit vector along z-axis),

$$\boldsymbol{J}(r,\phi,z;\omega) = q\,\delta(r)\delta(\phi)e^{-j\frac{\omega}{\beta c}z}\hat{z}_o; \tag{32}$$

For relativistic particles and small h the elecrtic field at the lowest frequence (n=0) is:

$$E_z = -\frac{qZ_0 h}{\pi a^3}\left[\delta\left(\frac{\omega}{c}-\beta_0'\right)+\delta\left(\frac{\omega}{c}+\beta_0'\right)\right]e^{-j\frac{\omega}{c}z} \tag{33}$$

From the definition Eq. (15), the longitudinal coupling impedence per unit length is found to be:

$$\frac{\partial Z(\omega)}{\partial z} = \frac{Z_0 h}{\pi a^3}\left[\delta\left(\frac{\omega}{c}-\beta_0'\right)+\delta\left(\frac{\omega}{c}+\beta_0'\right)\right] \tag{34}$$

and the longitudinal wake function per unit length:

$$\frac{\partial w(\tau)}{\partial z} = \frac{2Z_0 hc}{\pi a^3}\cos(\omega_0\tau) \tag{35}$$

with ω_0 given by Eq. (31).

8 Examples

We consider the simple case o a square waveguide, we get:

$$\frac{\partial w(\tau)}{\partial z} = 8\pi \frac{Z_0 \; c}{a^2} \; \frac{h}{a} \; \tanh\left(\frac{\pi}{2}\right) \left[\frac{\sinh(\pi)}{\pi} - 1\right]^{-1} \cos\left(2\pi \overline{f_1}\, \tau\right) \; H(\tau) =$$
$$= w_0\,(a,h) \; \cos\left(2\pi \overline{f_1}\, \tau\right) \; H(\tau). \tag{36}$$

Approximating the hyperbolic factors we have:

$$w_0\,(a,h) \; \simeq \frac{8\pi}{3} \frac{Z_0 \; c}{a^2} \; \frac{h}{a} = 9.6 \; 10^{-2}\pi^2 \left(\frac{h}{a^3}\right) ; [V/pCm] \tag{37}$$

The wake field has a resonant behaviour, and therefore can be expressed in terms of the single charge loss factor and in terms of the (R/Q)ratio.

$$w_0\,(a,h) \; \simeq \frac{\omega_r R_s}{Q_0} \; = 2k_0 \tag{38}$$

with

$$\omega_r = 2\pi \overline{f_1} \simeq c\sqrt{\frac{3}{ah}} \tag{39}$$

which gives

$$\frac{R_s}{Q_0} \simeq \frac{1,85\pi^2}{a} \left(\frac{h}{a}\right)^{3/2} ; [Ohm/m] \tag{40}$$