Relaxation Oscillations of the Synchrotron Motion Caused by Narrow-Band Impedance

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1 Outline

- SPEAR parameters
- Experimental data that characterizes the phenomenon
  - Spectrum analyzer
  - Streak camera
- Simulations that increase understanding
- Analytical model that explains this behavior

2 SPEAR Parameters

3 GeV e^- storage ring dedicated to synchrotron radiation

- Significant synchrotron radiation and associated damping
- Wide range of time scales involved in the phenomenon
  - Able to exploit this range in the analysis

Energy	**2.3GeV**	**HOM studied**	**Fundamental**
Natural Damping Time	**10 ms**	$\mathbf{f}_R = \mathbf{f}_{RF}$	**358.533 MHz**
Measured Damping Time	**5 ms (@2mA)**		**R**$_s$= **10 M**Ω
			Q $=$ **20000**
Relaxation oscillation	**<100Hz**	**>10 ms**	**>12800 turns**
f$_{so}$	**28.4 kHz**	**35** μ**s**	**45 turns**
Damping of HOM resonance	**56 kHz**	**17.8** μ**s**	**23 turns**
f$_0$	**1.28 MHz**	**0.78** μ**s**	**1 turn**
f$_{RF}$= **f**$_{HOM}$	**358.533 MHz**	**2.8 ns**	**1/280 turn**
Bunch spectrum (σ_τ^{-1}**)**	**2.8 GHz**	**57 ps**	**1/49 RF bucket**

3 Observations

- Longitudinal oscillations saturate
- Envelope oscillates at very low frequency (3 orders of magnitude smaller than ω_s)
- Previously experimentally observed and reported
  - PhotonFactory [Yamazaki 1983]
  - Surf 2 [Rakowsky 1985]
  - Elettra [Wrulich 1996]
- Previously studied theoretically
  - Suzuki and Yokoya [1982]
  - Krinsky [1985]
  - Nagaoka [1996]
- Characterized this behavior on largest available impedance, the fundamental mode of the idle cavity

4 Spectrum Analyzer

4.1 Data

- Observed signal from pickup in idle RF cavity
- Low envelope oscillation frequency $<$ 100 Hz (~ $\tau^{-1}_{radiation\,damping}$)
- Oscillation extends almost over entire region of instability
- Symmetry of growth time follows $R(\omega_{HOM}) = R\left(p\omega_0 + \omega_z\right)$
- Asymmetry of damping
  - Suggests complex damping mechanism
  - Explains frequency asymmetry
- Broadening of synchrotron frequency line:
  - $\Delta f_s \sim -15\%$
  - Variation of frequencies or frequency spread?

4.2 Results from Frequency Data

- The oscillation amplitude initially grows toward an attractor at ∞.
  - This is explained by linear theory
- Saturation occurs
- System dynamics change so that the motion evolves toward another attractor.
  - We call this cycle a **relaxation oscillation**.

These data give information only about the dipole moment of the bunch. We want to see its internal structure as well

5 Streak Camera

We were assisted by colleagues A. Fisher (SLAC), J. Hinkson and J. Byrd (LBNL), and A. Lumpkin (APS) in obtaining the beautiful pictures of the beam

- Correlates, as expected, with data from spectrum analyzer
- Bunch stays confined as a single macroparticle in the intial phase of relaxation cycle
- Loss of intensity during growth
  - Particles start to escape from the bunch and are seen distributed over time
- Large amplitude of oscillations ($\pm\frac{\pi}{2}$) i.e. 1/2 RF bucket size
  - Pendulum frequency decreases quadratically with amplitude
  - Spectrum analyzer data showed this 15% frequency shift over the relaxation cycle
- Growth limited by attractor at finite amplitude
- In most cases, images show that the bunch collapses to the center and begins a new cycle

- In the particular case of $\omega_z > \omega_s$
  - Damping rate very slow
  - 2^{nd} attractor appears at the bucket center
    - $\sim \pi$ out of phase with main body (or ‘initial attractor’)
    - starts growing while the ‘initial attractor’ is still damping

6 Simulations

- Photon intensity from optical port too low to get all the information desired from longitudinal distribution.
- Used a multiparticle tracking code to get more information about the behavior of the relaxation oscillation.
- Implements the standard synchrotron equations of motion, including quantum fluctuations and wake function

$$\begin{aligned} \phi_{n+1} &= \phi_n + 2\pi\eta\left(E_n - E_0\right)/E_0 \\ E_{n+1} &= E_n + eV_{RF}\cos\phi_{n+1} - U_{loss} + eV_W \end{aligned}$$

- The long range wake is calculated from turn to turn using propagators

$$\begin{pmatrix} W_{t+\Delta t} \\ W'_{t+\Delta t} \end{pmatrix} = \frac{e^{-\alpha\Delta t}}{\cos\phi_R} \begin{pmatrix} \cos\left(\Phi_{\Delta t} + \phi_R\right) & \frac{1}{\omega'_R}\sin\Phi_{\Delta t} \\ -\omega'_R\sin\Phi_{\Delta t} & \cos\left(\Phi_{\Delta t} - \phi_R\right) \end{pmatrix} \left[\begin{pmatrix} W_t \\ W'_t \end{pmatrix} + \begin{pmatrix} \alpha R_S n_t \\ -2\alpha^2 R_S n_t \end{pmatrix}\right] + \begin{pmatrix} \alpha R_S n_{t+\Delta t} \\ -2\alpha^2 R_S n_{t+\Delta t} \end{pmatrix}$$

- Bin size of 0.2σ
- 20000 particles in distribution, 10^5 turns for a relaxation cycle

6.1 Simulation Results

- Phase space plots
  - τ (time) on horizontal axis
  - δ (energy deviation) on vertical axis
  - Distribution rotates clockwise through 2π in one synchrotron period
- The code reproduced well the main observed behavior:
  - Instability thresholds
  - Relaxation oscillation amplitude and frequency
  - Diffusion from bunch
- These agreements gave confidence in the predictions from simulations
  - Diffusion of particles (filamentation) occurs from the front of the bunch in phase space
  - Filamentation occurs before main body has reached maximum amplitude
  - Decrease of charge density with time during growth
  - Particles spiraling toward the center of the bunch restart the cycle

7 Analytical Model

7.1 Goals

Based on the experimental data and computer simulations, the analytical model should explain:

- Instability thresholds given by linear theory
- Saturation mechanism of oscillation
- Diffusion, from the head of the bunch, as the amplitude grows
- Conditions for relaxation oscillation
  - Formation of second attractor
  - Damping mechanism
- For $\omega_z > \omega_s$, behavior of second attractor
  - $\sim \pi$ out of phase with initial attractor
  - growth as initial attractor damps

7.2 Equations of Motion

$$\dot{\delta} = \frac{\Delta\delta_n}{T_0} = \frac{\overbrace{eV_{RF}\sin\left(\omega_{RF}\tau + \varphi_S\right)}^{\text{Accelerating Voltage}} - \overbrace{U\left(\delta_n\right)}^{\text{Losses}} - \overbrace{eV_W\left(\tau\right)}^{\text{Wakefield}}}{E_0 T_0}$$

$$\dot{\tau} = \frac{\Delta\tau_n}{T_0} = \eta\delta$$

Rewrite this as a second order differential equation

$$\omega_{RF}\ddot{\tau} + \frac{\omega_{s_0}^2}{|\cos\varphi_S|}\sin\left(\omega_{RF}\tau + \varphi_S\right) = \frac{\omega_{s_0}^2}{eV_{RF}\,|\cos\varphi_S|}\left[U\left(\dot{\tau}\right) + eV_W\left(\tau\right)\right]$$

- This is the equation of a forced, biased pendulum
- Pendulum is a “soft” oscillator – frequency decreases with amplitude

7.3 Driving Term – Wakefield

7.3.1 Goal

- Calculate the wake voltage, the dynamic driving term, for a single particle of charge Ne
  - Source particle with charge Ne generates wake and arrives in cavity at time u
  - Test particle with charge e feels wake and arrives in cavity at time t
- Express this voltage in a closed analytical form that can be exploited for calculations
- Use results of experimental data and vastly different system time scales to derive an analytic expression

7.3.2 Procedure

- $(\sigma_\tau \ll 2\pi/\omega_R) \Rightarrow$ use impulse approximation for single pass of wakefield

  $$V(t-u) = NeW(t-u) = NeU(t-u)\, 2\alpha_R R_S e^{-\alpha_R(t-u)} \cos\omega_R(t-u)$$

  where $\alpha_R = \frac{\omega_R}{2Q}$
- Total wake becomes an infinite summation
- Source and test particles sample cavity only at discrete times
  - Define $t = nT_0 + \tau_t$,$u = kT_0 + \tau_u, \omega_R = p\omega_0 + \omega_z$
- Long range character of wake $\Longrightarrow$ approximate summation with integral
- The driving term is now analytically integrable, resulting in an infinite Fourier-Bessel series

  $$V(t) = 2\alpha_R R_S I\,\mathrm{Re}\left\{\sum_{p,m=-\infty}^{\infty} \frac{j^{p-m} J_p(r_t) J_m(r_u) e^{j(p\omega_{st}+m\omega_{su})t} e^{jp\phi_t + jm\phi_u}}{\alpha_r + j(m\omega_{su} - \omega_z)}\right\}$$

  where $r_t = \omega_R \bar{\tau}_t$ and $r_u = \omega_R \bar{\tau}_u$

This is still a very complex expression, so we exploit the slowly varying character, w.r.t. ω_s, of the amplitude, r_t, and phase, ϕ_t, of the oscillation.

7.4 Krylov-Bogoliubov-Mitropolskii (KBM) Averaging Method

- Driven harmonic oscillator

$$\ddot{\tau} + \omega_{s_0}^2 \tau = f_\tau(\tau, \dot{\tau})$$

- Define slowly varying functions $r(t)$, $\phi(t)$, from modified version of homogeneous solutions

$$\begin{aligned} \tau &= r(t) \cos(\omega_{s_0} t + \phi(t)) \\ \dot{\tau} &= -\omega_{s_0} r(t) \sin(\omega_{s_0} t + \phi(t)) \end{aligned}$$

- Solve the following equations

$$\begin{aligned} \frac{d\tau}{dt} &= \dot{\tau} \\ \frac{d\dot{\tau}}{dt} &= \ddot{\tau} = f_\tau(\tau, \dot{\tau}) - \omega_{s_0}^2 \tau \end{aligned}$$

- Invert to obtain differential equations for $r(t)$, $\phi(t)$

$$\begin{aligned} \dot{r} &= -\frac{1}{\omega_{s_0}} \sin(\omega_{s_0} t + \phi) f(r, \phi) \\ \dot{\phi} &= -\frac{1}{r\omega_{s_0}} \cos(\omega_{s_0} t + \phi) f(r, \phi) \end{aligned}$$

- Since r, ϕ, vary slowly, average over one synchrotron period (Fourier components)

$$\dot{\bar{r}} = -\frac{\omega_{s_0}}{2\pi}\int_{t-\frac{2\pi}{\omega_{s_0}}}^{t} \sin\left(\omega_{s_0}\tau + \phi\right) f\left(r, \phi\right) d\tau$$

$$\dot{\bar{\phi}} = -\frac{\omega_{s_0}}{2\pi r}\int_{t-\frac{2\pi}{\omega_{s_0}}}^{t} \cos\left(\omega_{s_0}\tau + \phi\right) f\left(r, \phi\right) d\tau$$

- Equations of motion include terms from wake, radiation damping, and pendulum equation

$$\dot{\bar{r}}_t = -\frac{1}{2\omega_{st}}\overbrace{F_{S1}\left(\bar{r}, \bar{\phi}\right)}^{\text{Wake}} - \overbrace{\alpha_{rad}\bar{r}_t}^{\text{Radiation}}$$

$$\dot{\bar{\phi}}_t = -\frac{1}{2\bar{r}_t\omega_{st}}\overbrace{F_{C1}\left(\bar{r}, \bar{\phi}\right)}^{\text{Wake}} - \overbrace{\frac{1}{16}\bar{r}_t^2\omega_{s_0}}^{\text{Pendulum}}$$

where $F_{C1}\left(\bar{r}, \bar{\phi}\right)$ and $F_{S1}\left(\bar{r}, \bar{\phi}\right)$ are the Fourier coefficients w.r.t. $\left(\omega_{s_0}\tau + \phi\right)$.

For a two particle model, the wake, and therefore its Fourier coefficients, depend on the properties of both particles. These coefficients, for a system with a source particle (r_u, ϕ_u) carrying current, I, acting on a test particle (r_t, ϕ_t) with infinitesimal charge, are

$$F_{S1} = -A \sum_{k=1}^{\infty} J_k(r_u) \left[J_{k-1}(r_t) + J_{k+1}(r_t) \right] \times \left[\left(a_k^- - a_k^+ \right) \cos(k\Delta\phi) - \left(b_k^- - b_k^+ \right) \sin(k\Delta\phi) \right]$$

$$F_{C1} = A \left\{ 2b_0^+ J_0(r_u) J_1(r_t) + \sum_{k=1}^{\infty} J_k(r_u) \left[J_{k-1}(r_t) - J_{k+1}(r_t) \right] \times \left[\left(b_k^- - b_k^+ \right) \cos(k\Delta\phi) + \left(a_k^- - a_k^+ \right) \sin(k\Delta\phi) \right] \right\}$$

where $A = (2\alpha_R R_S I) \frac{\omega_{s_0}^2}{V_{RF}|\cos\phi_s|}$, $\Delta\phi = \phi_t - \phi_u$and

$$a_k^{\pm} = \frac{\alpha_R}{\alpha_R^2 + (k\omega_{su} \pm \omega_z)^2}; \quad b_k^{\pm} = \frac{(k\omega_{su} \pm \omega_z)}{\alpha_R^2 + (k\omega_{su} \pm \omega_z)^2}$$

7.5 Analysis

- Infinite sum that depends on the oscillation amplitude of the particles, $\bar{r}$
- From data, $\bar{r}_{\rm max} \sim \pi/2$, so the series converges quickly due to the Bessel functions
- $a_k \sim \mathrm{Re}\left\{Z\left(\omega\right)\right\};\ b_k \sim \mathrm{Im}\left\{Z\left(\omega\right)\right\}$
- Antisymmetric with respect to $\pm\omega_z$
  - One side of resonance provides damping
  - Other side of resonance provides growth
- A very good understanding comes from keeping only the lowest terms
- These forces have a strong dipole characteristic

7.5.1 Linear theory

$(r_t, \phi_t) = (r_u, \phi_u)$ and use small amplitude expansion of $J_n(r)$

- For $\omega_z = \omega_{s_0}$

$$\begin{aligned} F_{S1} &\approx -Aa_1^- J_1(r)\, J_0(r) \\ &\approx -\frac{A}{\alpha_R}\frac{r}{2} \\ \frac{\dot{r}}{r} &\approx \frac{\omega_R}{\omega_{RF}} \frac{\omega_{s_0} R_S I}{2V_{RF}\left|\cos\phi_s\right|} - \alpha_{rad} \end{aligned}$$

- Shows odd symmetry of growth/damping and frequency shift with respect to the fractional part of the resonator frequency, ω_z

7.5.2 Saturation

$(r_t, \phi_t) = (r_u, \phi_u)$ but now evaluate non-linearities in sum

- Decrease in amplitude of sum with increasing argument gives saturation mechanism
- Data shows saturation at earlier level, consistent with particle loss from main bunch

7.5.3 Diffusion

- Fluctuations in the energy of individual particles cause them to wander from the macroparticle
  - Stability $\Longrightarrow$ restoring forces on particles bring them back to the macroparticle
  - Instability $\Longrightarrow$ restoring forces on particles drive them away from macroparticle
- Move to rotating coordinate frame in which $\phi_u = 0$
- Consider the case of the source particle, (r_u, ϕ_u), carrying all the charge and examine the behavior of a test particle, (r_t, ϕ_t), at an arbitrary location in phase space.
- In this frame, calculate if the separation between (r_t, ϕ_t) and (r_u, ϕ_u) is increasing or decreasing

At small amplitudes

- $F_{S1} \Rightarrow$ radial restoring force
- $F_{C1} \Rightarrow$ azimuthal restoring force
- $\Rightarrow$ macroparticle is stable

At large amplitudes

- Characteristics of radial forces are unchanged
- Azimuthal force now dominated by pendulum effect
  - Test particle that falls behind macroparticle sees less $\dot{\bar{r}}$, drops toward the origin, gains $\dot{\bar{\phi}}$, and returns to region of larger $\dot{\bar{r}}$.
  - Test particle that moves ahead also sees less $\dot{\bar{r}}$, drops toward the origin, gains even more $\dot{\bar{\phi}}$, and exits macroparticle from the front.
- As test particle rotates away from macroparticle, the wake forces oscillate between attraction and repulsion and lose their net strength. Radiation damping brings the test particle back toward the origin.

This change from attraction to the macroparticle to repulsion from it shows the change in dynamics of the system needed to form the second attractor and create the necessary conditions for the relaxation oscillation.

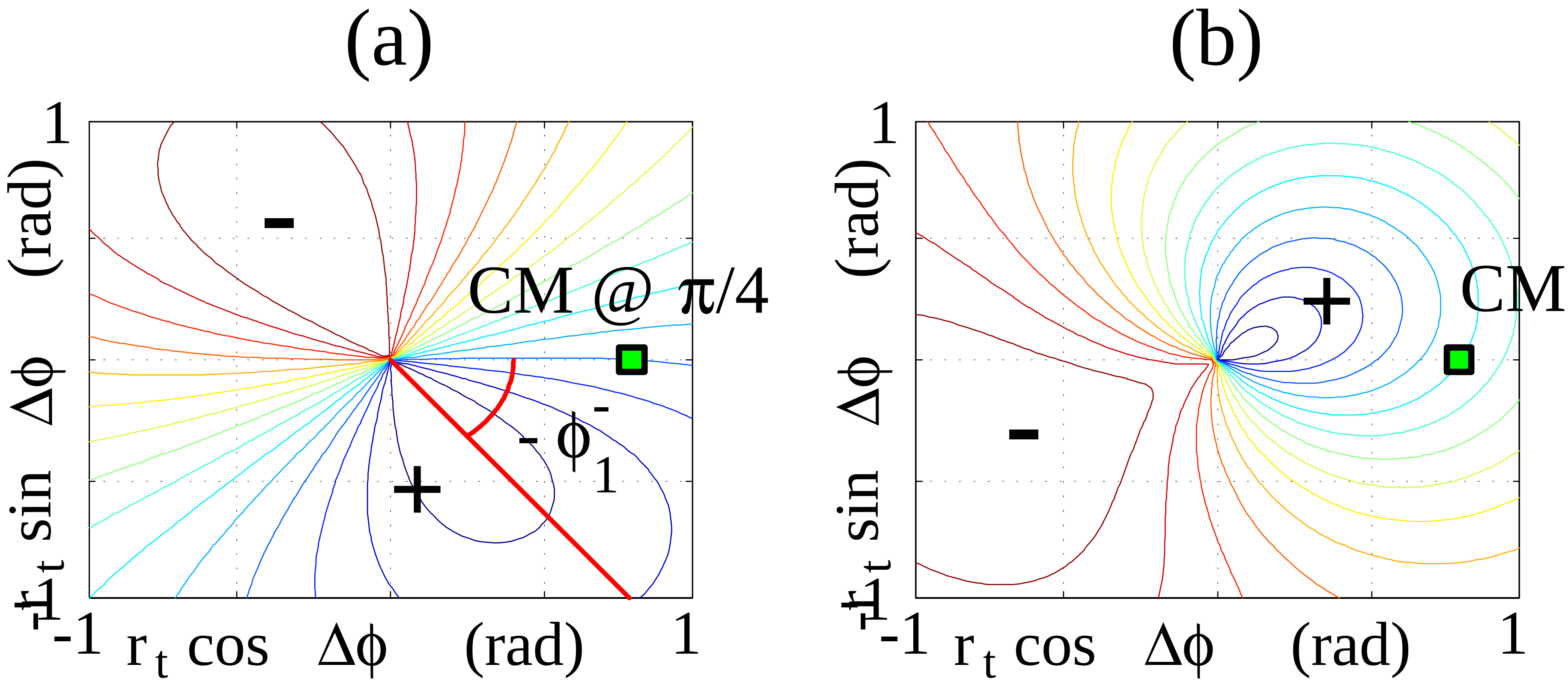
7.5.4 Location of second attractor near origin

- KBM term for $\dot{\bar{\phi}}$ has a factor r^{-1}
  - $\dot{\bar{\phi}}$ can assume any value near the origin
  - There exists locus of points near origin phase locked to macroparticle
  - A test particle on this locus will feel the wake of the macroparticle
- Second attractor moves slowly $\Rightarrow$ close to fixed point $(\dot{\bar{r}} = 0)$ about $\pi/2$ ahead of macroparticle
- As particles accumulate at this second attractor its charge increases
  - Its charge contributes to its own growth
    - Its amplitude grows
  - To stay nearly fixed, it needs to see more damping from original attractor
    - Its angle $\bar{\phi}$ increases toward π
  - Two centers exert damping force on each other
  - Initial attractor damps

Relaxation cycle is then flow of particles between two centers, with cycle frequency $1/2$ that of observed signal.

7.5.5 Asymmetry of damping and observation of second attractor

- When $\omega_z > \omega_s$, F_{S1} and F_{C1} rotate clockwise with two results
- Test particles pass through higher regions of F_{S1} as they try to escape from the front of the bunch
  $\Rightarrow$ longer time needed for macroparticle to decay
- Node of $\dot{r} = 0$, $\omega_{st} = \omega_{su}$ is shifted closer to π
  $\Rightarrow$ second attractor close to π out of phase with initial attractor



$\omega_z > \omega_s$: (a) Growth from wake; (b) Frequency shift from wake

8 Conclusions

- Experimental observations give complete characterization of phenomenon
- Simulations
  - Good agreement with experiments (low frequency behavior, filamentation)
  - Additional predictive power of direction of filamentation
- Analytic model
  - Precise derivation of long term wake potential
  - Theoretical explanation of phenomenon explains
    - Instability theory of linear model
    - Saturation mechanism of synchrotron oscillation
    - Diffusion mechanism
    - Conditions for relaxation oscillation
    - Creation and location of second attractor
    - Asymmetry of damping mechanism