Linear Collider Collaboration Tech Notes

Consistent Yokoya-Chen Approximation
to Beamstrahlung
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Abstract: I reconsider the Yokoya-Chen approximate evolution equation for beamstrahlung and modify it slightly to generate simple, consistent analytic approximations for the electron and photon energy spectra. I compare these approximations to previous ones, and to simulation data.

For many purposes, one needs a simple parametrization of the energy distributions of the colliding electrons and photons at linear colliders. These distributions result from the beam-beam interaction and thus are complicated to compute directly. In 1989, Kaoru Yokoya and Pisin Chen [1] introduced a clever approximation scheme which allows one to obtain a fairly simple parametrization of these distributions. For various reasons, they were not particularly happy with the results and later introduced a number of elaborations of their formalism [1, 2, 3]. In this note, I will argue that their original idea, carried out consistently, produces results which are as good as any other simple parametrization and quite adequate for a quantitative description.

Let $\psi_e(x)$ and $\psi_\gamma(x)$ represent the electron and photon distributions resulting from beam-strahlung. Here x is the fraction of the nominal beam energy carried by the electron or photon, $0 < x \leq 1$. If we disregard electrons from pair creation, the electron distribution should be normalized to

$$1 = \int dx \psi_e(x) \ . \tag{1}$$

Since the momentum radiated by the electrons is carried by the photons, the distributions should also satisfy the sum rule

$$1 = \int dx x (\psi_e(x) + \psi_\gamma(x)) \ . \tag{2}$$

As the electron and positron bunches pass through each other, the electron distribution evolves according to an equation

$$\frac{\partial \psi_e}{\partial t} = -\nu(x)\psi_e(x,t) + \int_x^1 dx' F(x,x')\psi_e(x',t) \ , \tag{3}$$

where F(x,x') is determined by the Sokolov-Ternov synchrotron radiation spectrum in an effective magnetic field determined by the accelerator parameters. The coefficient $\nu(x)$ is the average number of photons radiated per unit time at the given value of x. Eq. (3) cannot be solved analytically, and in any case the solution must be averaged over the time-dependence of the bunch crossing. Clearly a judicious approximation scheme is needed.

Electron Spectrum

Yokoya and Chen [1] suggested replacing (3) by a simpler evolution equation. Let ν is taken to represent the classical radiation probability, let Υ be the quantum parameter of synchrotron radiation, and let N_γ be the nominal number of photons emitted per electron in beamstrahlung; I will give formulae for these below. Then Yokoya and Chen wrote

$$\frac{\partial \psi_e}{\partial t} = -\nu \psi_e(x,t) + \nu \int_x^1 dx' \frac{\kappa}{xx'} g(\eta - \eta') \psi_e(x',t) , \tag{4}$$

in which $\kappa = 2/3\Upsilon$ and $\eta = \kappa/x$. This choice was supposed to turn the equation into a translation-invariant equation in η, which could then be solved by Laplace transformation. The resulting function $\psi(x,t)$ could then be evaluated for $\nu t = N_\gamma$. I will refer to this result as the YC0 approximation.

Yokoya and Chen were not satisfied with the comparison between YC0 and simulation data. They gave an alternative prescription in which the function $\psi(x,t)$ would be evaluated for νt equal to a function of x which varied from the classical result for N at large x to the quantum result at small x. In a later paper [2], Chen introduced an even more elaborate approximation in which he attempted to take account of the average over the time-dependent overlap of the bunches. I will refer to these approximations as YC1 and C2, respectively.

Many people who have tried to use these functions have observed that none of the three approximations satisfy the sum rule (1). This is awkward for Monte Carlo simulations of physics processes, for which one wishes to normalize event rates to the given luminosity. It is no surprise that YC1 and C2 should violate the normalization condition, given that they tamper with $\psi(x)$ as a function of x. But the failure of the normalization condition for YC0 indicates that something was wrong with the original approximation.

The problem can be fixed simply by replacing (4) by the evolution equation

$$\frac{\partial \psi_e}{\partial t} = -\nu \psi_e(x,t) + \nu \int_x^1 dx' \frac{\kappa}{x^2} g(\eta - \eta') \psi_e(x',t) . \tag{5}$$

It is easy to check that this equation exactly preserves the probability sum rule. This equation can then be solved by the method of Yokoya and Chen, producing the solution

$$\psi(x) = e^{-N} \left[\delta(x-1) + \frac{e^{-\kappa(1-x)/x}}{x(1-x)} h(y) \right] , \tag{6}$$

where $N = \nu t$,

$$y = N(\kappa(1-x)/x)^{1/3} , \tag{7}$$

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and $h(x)$ is a function that appears in the Yokoya-Chen paper:

$$h(y) = \sum_{n=1}^{\infty} \frac{y^n}{n!, \, (n/3)} \; . \tag{8}$$

This function has the asymptotic expansion

$$h(y) = \left(\frac{3z}{8\pi}\right)^{1/2} e^{4z} \left[1 - \frac{35}{288}\frac{1}{z} - \frac{1295}{16588}\frac{1}{z^2} + \cdots\right] \tag{9}$$

with $z = (y/3)^{3/4}$. The function $h(x)$ can be computed quite accurately by using (8) and (9) in their respective regions. (Yokoya and Chen give a simple approximate expression for $h(y)$; this misses badly and should not be trusted.)

To compare these various functions to simulation data, let me first quote the formulae which connect accelerator parameters to synchrotron radiation parameters. Here approximations are also being made. I follow Chen's prescriptions from [3]. Let the horizontal and vertical disruption parameters D_x and D_y (corresponding to beam sizes $\sigma_x/\sigma_y \gg 1$) be given by

$$D_a = \frac{2\mathcal{N}r_e\sigma_z}{\gamma\sigma_a(\sigma_x + \sigma_y)} \tag{10}$$

for $a = x, y$, $\gamma = E/m_e$ for the beam, $\mathcal{N}$ the number of particles per bunch, r_e the classical electron radius, and σ_z the longitudinal beam size. The corresponding luminosity enhancement parameters are

$$H_a = 1 + D_a^{1/4}\frac{D_a^3}{1 + D_a^3}\left(\log(\sqrt{D_a} + 2\log(0.8\beta_a/\sigma_z))\right) \; , \tag{11}$$

where β_a are the horizontal and vertical β functions at the interaction point. Let [sic]

$$\overline{\sigma}_x = \sigma_x H_x^{-1/2} \; , \qquad \overline{\sigma}_y = \sigma_y H_y^{-1/3} \; . \tag{12}$$

Then the effective value of the synchrotron radiation parameter should be given by

$$\Upsilon = \frac{5}{6}\frac{r_e^2\gamma\mathcal{N}}{\alpha\sigma_z(\overline{\sigma}_x + \overline{\sigma}_y)} \; . \tag{13}$$

The classical and quantum radiation rates are given by

$$\nu_{\rm cl} = \frac{5}{2\sqrt{3}}\frac{\alpha^2}{r_e\gamma}\Upsilon \; , \qquad \nu_\gamma = \nu_{\rm cl}[1 + \Upsilon^{2/3}]^{-1/2} \; . \tag{14}$$

Finally

$$N_{\rm cl} = \sqrt{3}\sigma_z\nu_{\rm cl} \; , \qquad N_\gamma = \sqrt{3}\sigma_z\nu_\gamma \; . \tag{15}$$

For the August 1998 parameter sets, I find:

NLC=500-B $N_\gamma = 1.11$ $\Upsilon = 0.104$
NLC=1000-B $N_\gamma = 1.43$ $\Upsilon = 0.299$

The various approximations above can be compared to the results from computer simulation of beam collision, including beamstrahlung and disruption. The simulations used below were generated by Kathy Thompson using Daniel Schulte's Guinea Pig simulation program, with initial state radiation turned off, and with zero intrinsic beam energy spread.

For the purpose of physics simulations, one is interested in the energies of the electrons and positrons that collide and annihilate. Therefore, I will compare the various distributions to the energy distributions for colliding particles. I will not take account of the correlations between the energies of colliding particles; I will simply compare to the basic energy distribution. In the figures that follow, the histogram will represent the simulation results, normalized to an area of 1. The blue curve will be the approximation (6) evaluated with $N = N_\gamma$. Henceforth, I will call this approximation P1. The green curve will be the same approximation with $N = N_\gamma/2$, henceforth, P2. This change reflects the idea that particles that collide have not given off their complete radiation. The red curves are the YC results; YC1 is always higher than YC0. The magenta curve is the approximation C2 [5]. The delta function at $x = 1$ is not shown. The approximations P1 and P2 are guaranteed to be normalized to 1 when the delta function is included. The approximations YC0, YC1, C2 are plotted as given in [1, 2] and are not renormalized. In all cases, the total normalization of YC1 is too high by about 10%, and the of YC0 and C2 is too low.

Figures 1, 2, and 3 show the comparison for the NLC-500B. Figure 1 shows the overall comparison on a linear scale for $\psi_e(x)$. Figure 2 uses a logarithmic scale to show the low energy tail. Figure 3 is a blow-up of the highest-energy region. P2 and C2 provide the best approximation over the broad range, though, in detail, they are low in the highest-energy bins and high at lower energies. They do reproduce the falloff at low energies seen in the simulation data. Changing the value of N used to compute P2 does not noticeably improve the fit.

Figures 4, 5, and 6 show the same comparison for the NLC-1000B. The various approximations succeed and fail in a quite similar fashion.

Photon Spectrum

As beamstrahlung degrades the electrons in the energy, it also generates the photon distribution $\psi_\gamma(x)$. This distributions should be computed from an evolution equation similar to (3), in such a way that the momentum sum rule (2) is preserved. It is not so obvious how to generate the photons consistently with the electrons in the approximation schemes YC1 or C2. A rather complex treatment is given in [2], and the formulae are repeated in a somewhat simpler form in [3].

But, if we use the equation (5) without further approximations, it is clear how to generate

a consistent photon spectrum. We simply write the corresponding photon equation

$$\frac{\partial \psi_\gamma}{\partial t} = \nu \int_x^1 dx' \frac{\kappa}{(x'-x)^2} \, g(\frac{\kappa}{x'-x} - \frac{\kappa}{x'}) \, \psi_e(x',t) \ , \tag{16}$$

and integrate with the known function $\psi_e(x,t)$. It is easy to see that the resulting photon spectrum is guaranteed to satisfy (2).

There is a problem with this approach, however. The integral with (6) is complicated and cannot be done analytically. I was not able to find an exact representation that is reasonable to compute. However, one might notice that most of the beamstrahlung radiation actually comes from the delta function term in (6), for which the integral is trivial. For the rest, the photons are mainly produced at small x and the approximation that $x \ll x'$ is a reasonable one. A simple treatment is to approximate the whole electron distribution as a normalized delta function at $x = 1$. The resulting photon distribution is very simple:

$$\psi_\gamma(x) = N \frac{\kappa^{1/3}}{, \ (1/3)x^{2/3}(1-x)^{4/3}} e^{-\kappa x/(1-x)} \ , \tag{17}$$

with $N = \nu t$ and κ as above. The approximation generates a photon spectrum which is slightly too hard for consistency; however, the discrepancy is small. For the NLC-500B design and $N = N_\gamma$, I find a discrepancy of less than 0.1%, while for the NLC-1000B design, I find that there is a 1% excess over the sum rule (2). For a 1500 GeV parameter set [6], which gives $N_\gamma = 1.8$ and 22% of the total momentum in photons, I find a 4% excess. The more reasonable choice of $N = N_\gamma/2$ gives smaller discrepancies.

I will label the approximation (17) with $N = N_\gamma$ as PG1 and the same approximation with $N = N_\gamma/2$ as PG2. I will refer to Chen's approximation from [3] as CP2. These distributions can be compared to the energy distributions for colliding photons produced by the Guinea Pig simulations described above. The simulation data are normalized to unit area. For comparison to these data, I have divided the analytic approximations by the factor

$$\sqrt{N(\gamma\gamma)/N(e^+e^-)} \ , \tag{18}$$

where $N(e^+e^-)$ and $N(\gamma\gamma)$ are the numbers of events in the files generated by Guinea Pig which sample the e^+e^- and $\gamma\gamma$ luminosity functions. These two events numbers are expected to correspond to the physically correct ratio of luminosities.

Figures 7 and 8 show this comparison for the NLC-500B design. The blue and green curves refer to PG1 and PG2, as before; the magenta curve refers to CP2. The approximations PG2 and CP2 are actually in good agreement, and both have approximately the correct normalization, while PG1 is naturally much larger. Figure 8 shows that the analytic approximations have a longer tail toward high energies than the simulation data. Figures 9 and 10 show the same comparisons for the NLC-1000B design.

Conclusions

Though none of the various approximations discussed gives an excellent fit to the electron and photon spectra from beamstrahlung, the approximation which begins from the evolution equations (5) and (16), and evolves for a time $\nu t = N_\gamma/2$ gives a plausible representation of the simulation data and automatically respects the sum rules (1) and (2). I thus recommend the formulae (6) and (17), with $N = N_\gamma/2$, as a reasonable set of analytic formulae to approximate beamstrahlung for physics simulations.

Appendix: Initial State Radiation

At the same time that we must consider beamstrahlung, we should consider prescriptions for initial state radiation. In this appendix, I would like to give what I consider the best compromises between simplicity and accuracy.

The distribution of electrons and photons after initial state radiation may be computed from the Gribov-Lipatov equations [7], the analogue (and predecessor) for QED of the Altarelli-Parisi equations. To leading order, the distributions of electron and photon energies for an electron beam of fixed energy are given by the solution to

$$
\begin{aligned}
D_e(x,s) &= \delta(x-1) + \int_{m_e^2}^{s} \frac{ds'}{s'} \frac{\alpha(s')}{2\pi} \int_x^1 \frac{dz}{z} P_{e \to e}(z) D_e(\frac{x}{z}, s') \\
D_\gamma(x,s) &= \int_{m_e^2}^{s} \frac{ds'}{s'} \frac{\alpha(s')}{2\pi} \int_x^1 \frac{dz}{z} P_{e \to \gamma}(z) D_e(\frac{x}{z}, s')
\end{aligned}
\tag{19}
$$

where

$$
\begin{aligned}
P_{e \to e}(z) &= \left. \frac{1+z^2}{(1-z)} \right|_+ + \frac{3}{2}\delta(z-1) \\
P_{e \to \gamma}(z) &= \frac{1+(1-z)^2}{z} \ .
\end{aligned}
\tag{20}
$$

These equations sum up the rates of multiple collinear photon emission from the incoming electron. A pedagogical derivation of these equations, and the definition of $|_+$, can be found in [8].

Kuraev and Fadin (KF) [9] presented an approximate analytic solution to this equation. In addition, they argued that, if one modifies the solution by the substitution

$$
2\log(2E_{\text{beam}}/m_e) \quad \to \quad \beta = 2\log(2E_{\text{beam}}/m_e) - 1 \ ,
\tag{21}
$$

the resulting function takes non-logarithmic corrections into account correction to about 1% accuracy. Some effects of α running are omitted. Subsequently, Nicrosini and Trentadue [10] showed how to introduced additional modifications to obtain 0.1% accuracy for the Z^0 line shape.

To my mind, few-percent accuracy is good enough for the initial linear collider studies. So the solution of KF is quite acceptable. I would recommend, however, using a simpler approximate solution of the Gribov-Lipatov equation written down by Skrzypek and Jadach (SJ) [11]:

$$
\begin{aligned}
D_e(x) \;=\; & \frac{1}{2}\eta(1-x)^{\eta/2-1} \\
& \cdot\; (1+\frac{1}{2}\eta)e^{-(1/8)(\eta+(\pi^2/6-1)\eta^2)}\left[\frac{1}{2}(1+x^2)-\frac{\eta}{8}(\frac{1}{2}(1+3x^2)\log x+(1-x)^2\right] ,(22)
\end{aligned}
$$

where η takes a part of the α running into account,

$$
\eta = -6\log(1-\frac{1}{6}\beta) \; . \tag{23}
$$

The KF and SJ formulae are compared in Figure 11. They differ by 2%; One cannot really go wrong. The first line of (22) is a simple normalized function that does the leading part of the resummation. This is compared to the other two functions in the figure.

For a polarized electron beam, the electrons preserve their polarization after radiation (up to effects of size $\alpha/\pi \sim 0.1\%$, without any logarithmic enhancement). However, a polarized electron radiates photons of both polarizations, and it is sometimes necessary to take this into account. This can be done easily from the Gribov-Lipatov equations, by separating the kernel in the photon equation (for an e_R^- beam) into a part which produces right-handed photons and a part which produces left-handed photons,

$$
P_{\to\gamma_R}(z) = \frac{1}{z} \; , \qquad P_{\to\gamma_L}(z) = \frac{(1-z)^2}{z} \; . \tag{24}
$$

I am not aware of any simple approximate solutions to the Gribov-Lipatov equations for the photon distribution which have been given in the literature. One very simple approximation is the analogue of the one made above for beamstrahlung: Take the electron distribution radiating the photons to be a delta function at $x = 1$. Then we obtain ('simple GL'):

$$
D_{\gamma_R}(x) = \frac{1}{4}\eta\frac{1}{x} \; , \qquad D_{\gamma_L}(x) = \frac{1}{4}\eta\frac{(1-x)^2}{x} \; . \tag{25}
$$

This has the problem (not a very serious one) that $D(x)$ goes to a constant, nonzero value at $x = 1$. To cure this problem, consider the effect of radiating from the electron distribution given by the first line of (22). The solution for $D_{\gamma_R}(x)$ is fairly simple:

$$
D_{\gamma_R}(x) = \frac{1}{2}\frac{\left(1-(1-x)^{\eta/2}\right)}{-\log(1-x)}\frac{1}{x} \; . \tag{26}
$$

For the left-handed photons, the obvious approximation is

$$
D_{\gamma_L}(x) = \frac{1}{2}\frac{\left(1-(1-x)^{\eta/2}\right)}{-\log(1-x)}\frac{(1-x)^2}{x} \; . \tag{27}
$$

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I will call (26),(27) the approximation 'improved GL'.

For the radiation of a single photon from an electron at $x = 1$, there is an improvement of the leading-log formula, due to Brodsky, Kinoshita, and Terazawa (BKT) [12]. Several authors used the BKT formalism to derive distributions of the W in the electron, which necessarily brings in polarization. I have used the BKT approximations to generate the polarized photon distributions from the electron; unfortunately, $D_L(x)$ goes very slightly negative at large x. It is possible to introduce an 'improved BKT' which avoids this problem. These approximations are compared to the solutions of the Gribov Lipatov equation in Figures 12 and 13. Curiously, the improved-GL and the improved BKT approximations agree to within about 5% over most of the range of x.

I advocate using SJ, (22), for the electron distribution from initial state radiation, and using improved-GL, (26) and (27), for the photon distribution.

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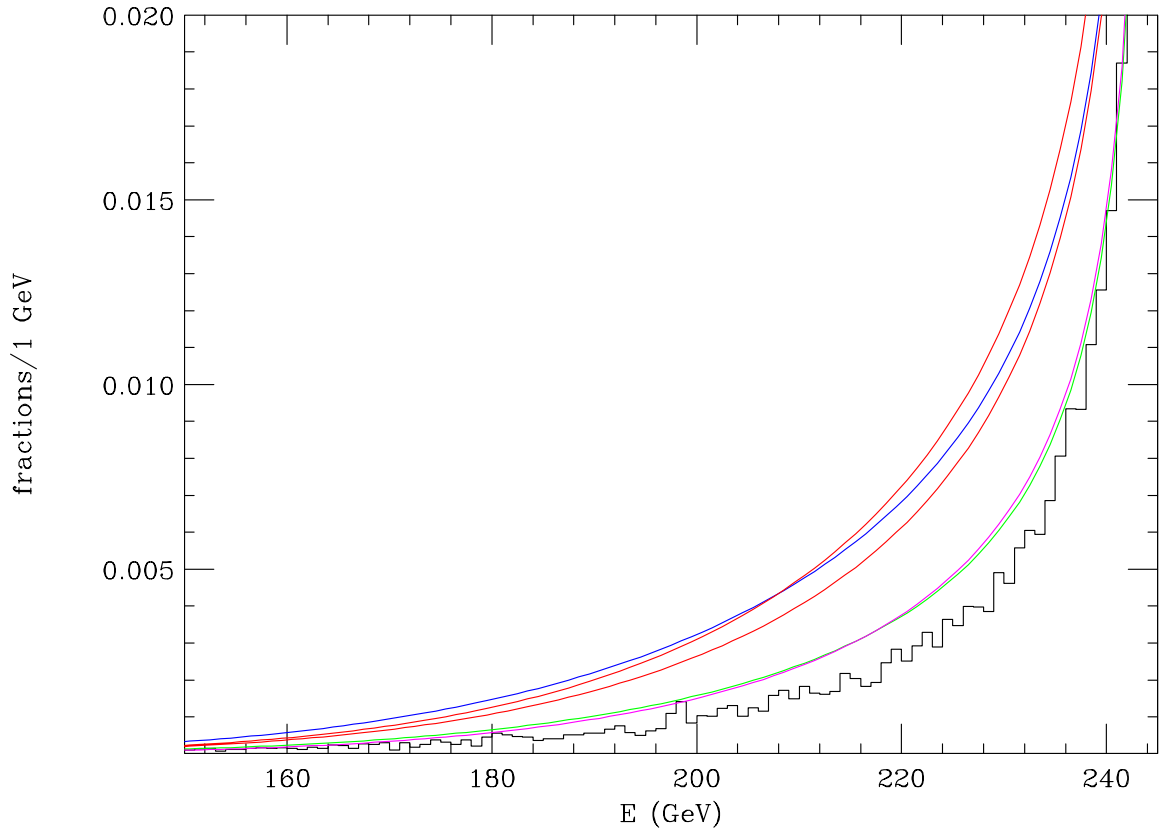
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Figure 1: Comparison of Guinea Pig simulation data on the electron energy spectrum with various analytic approximations for the NLC-500B design parameters: blue-P1, green-P2, red-YC1 over YC0, magenta-C2.
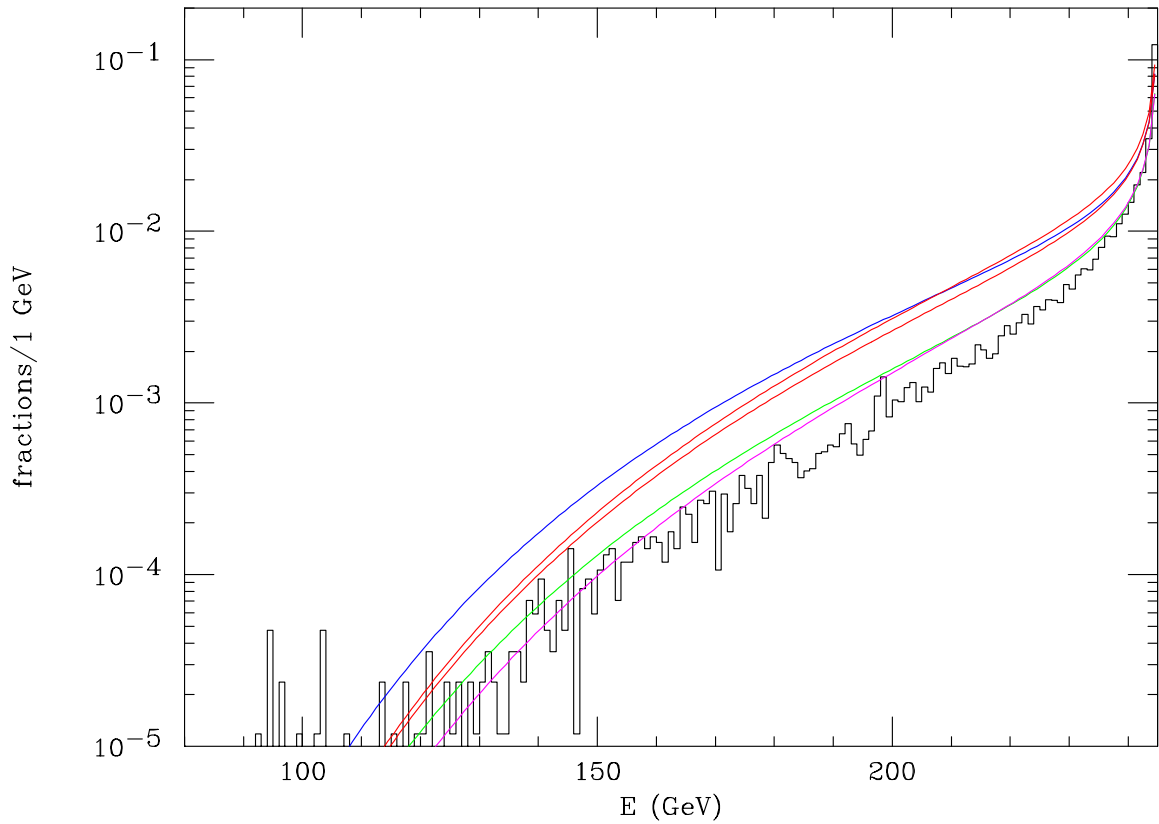
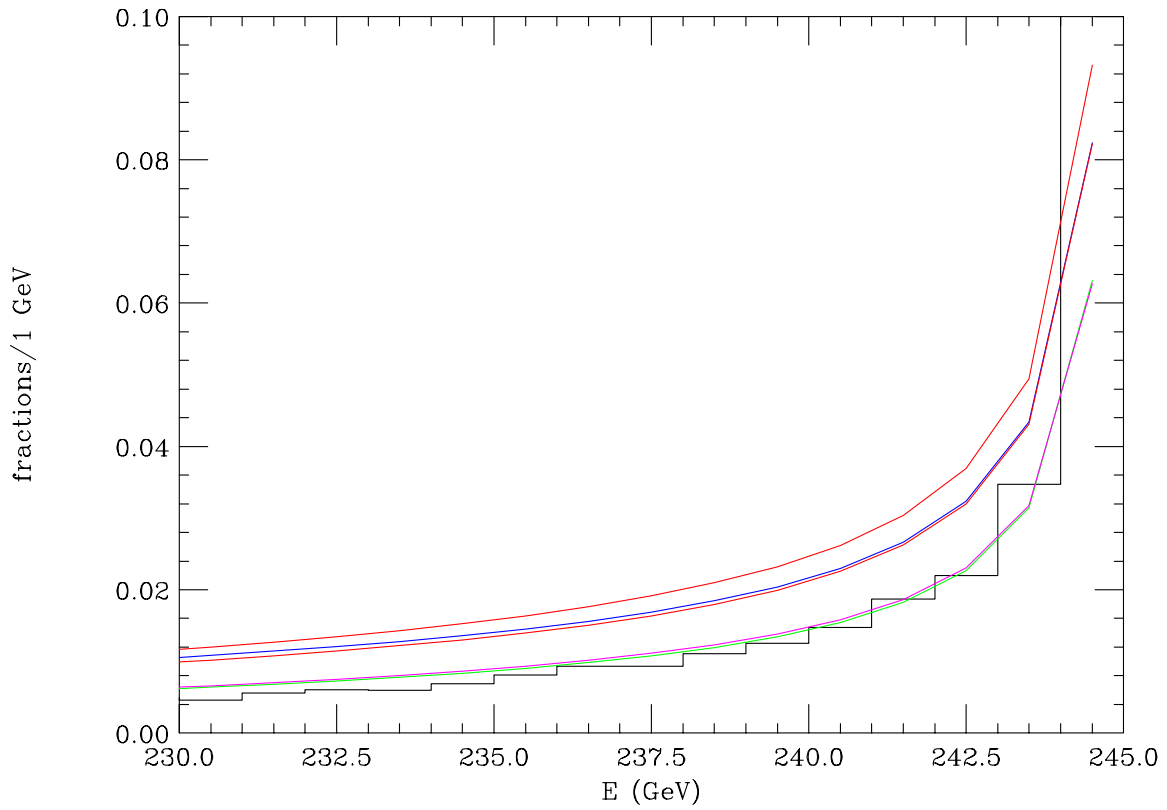Figure 2: Figure 1 with a logarithmic scale, showing the low-energy, low-x tail of the distribution.

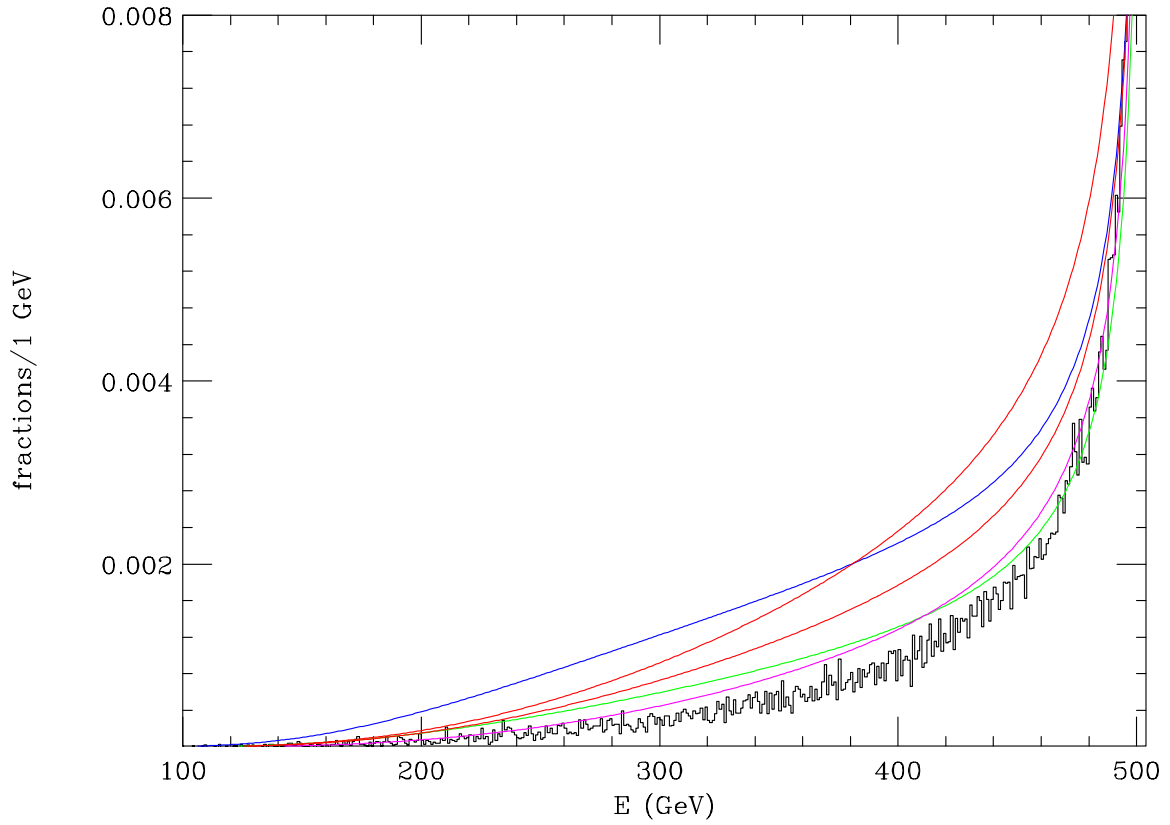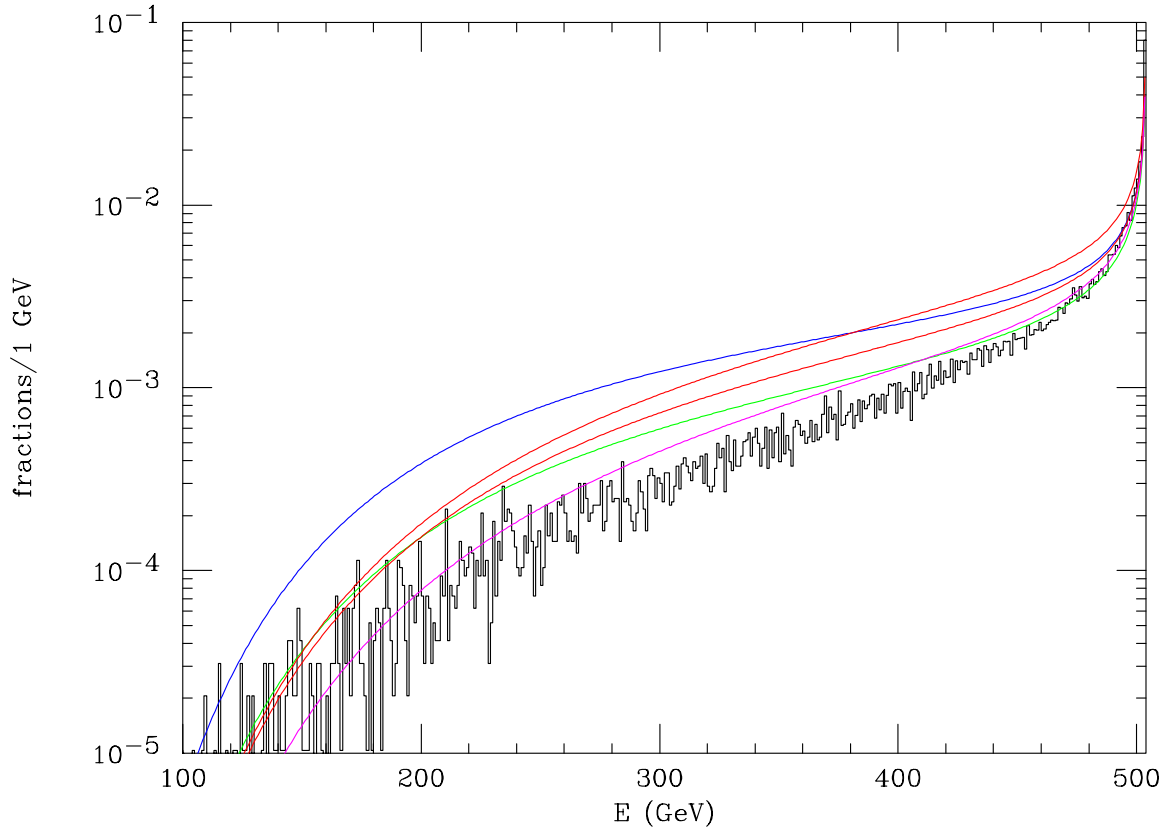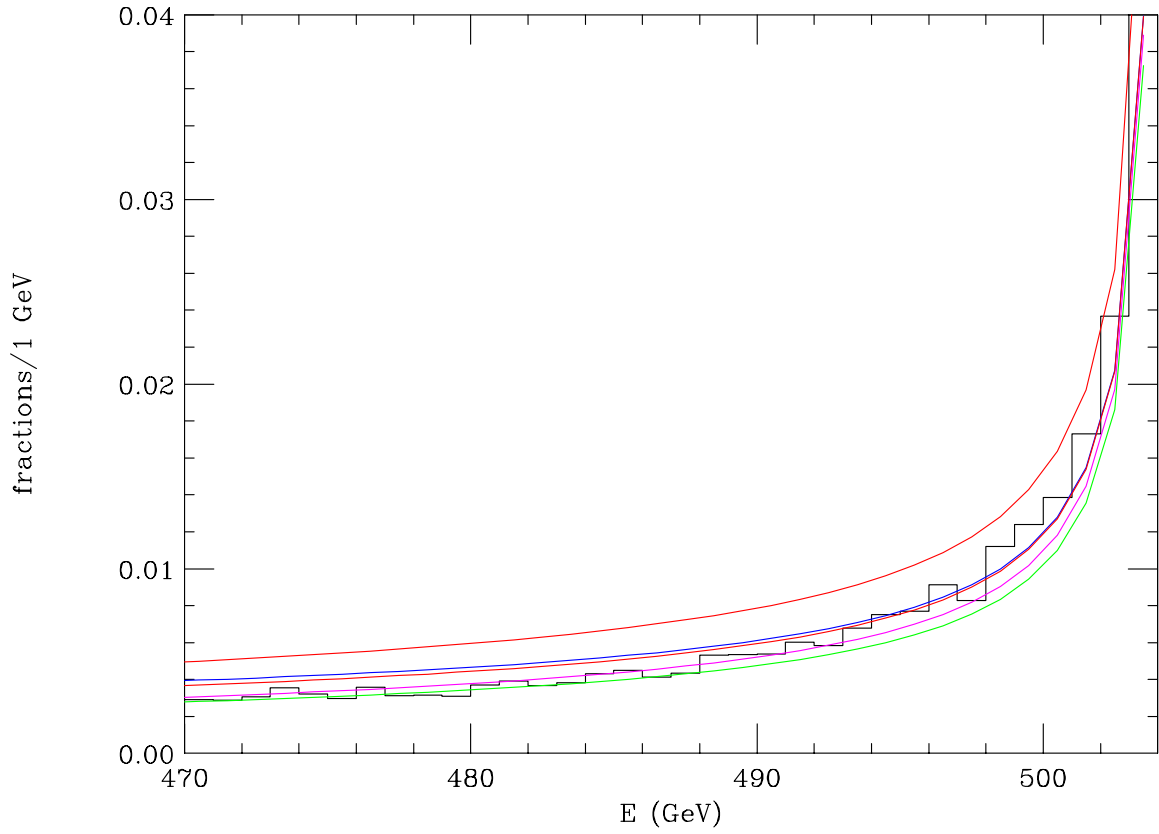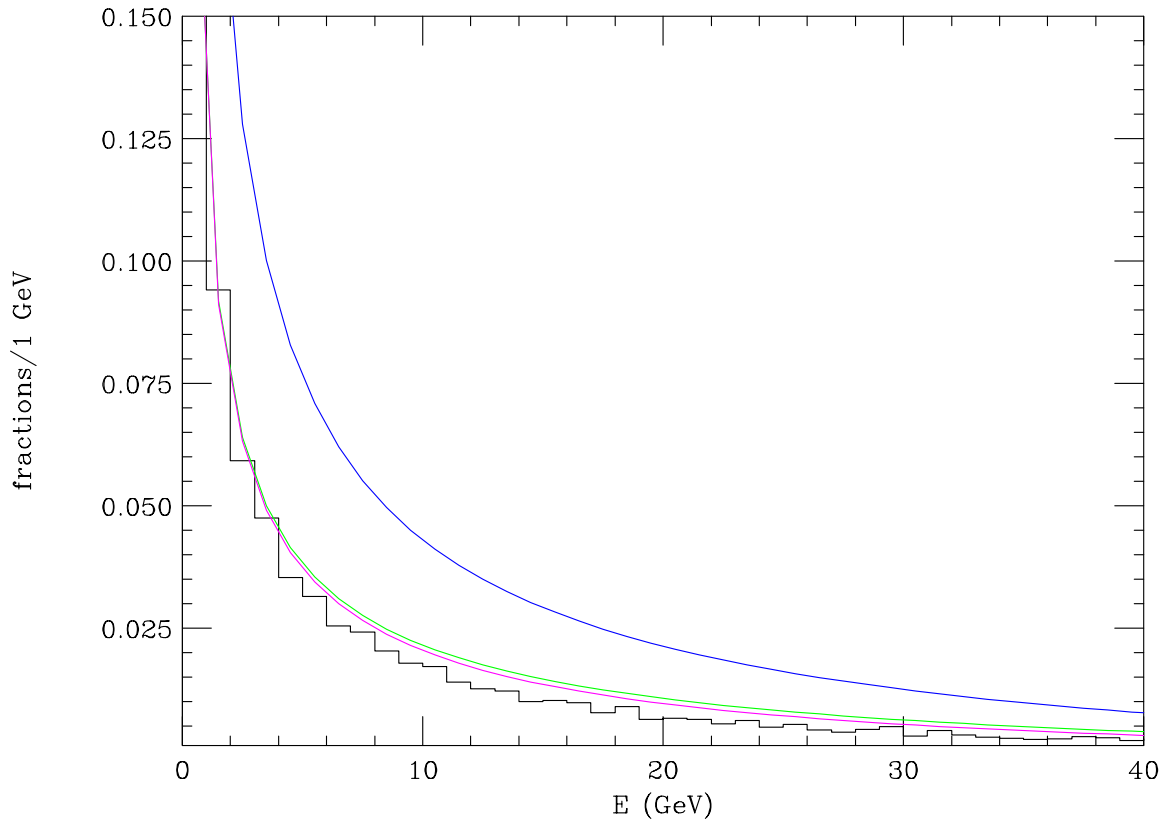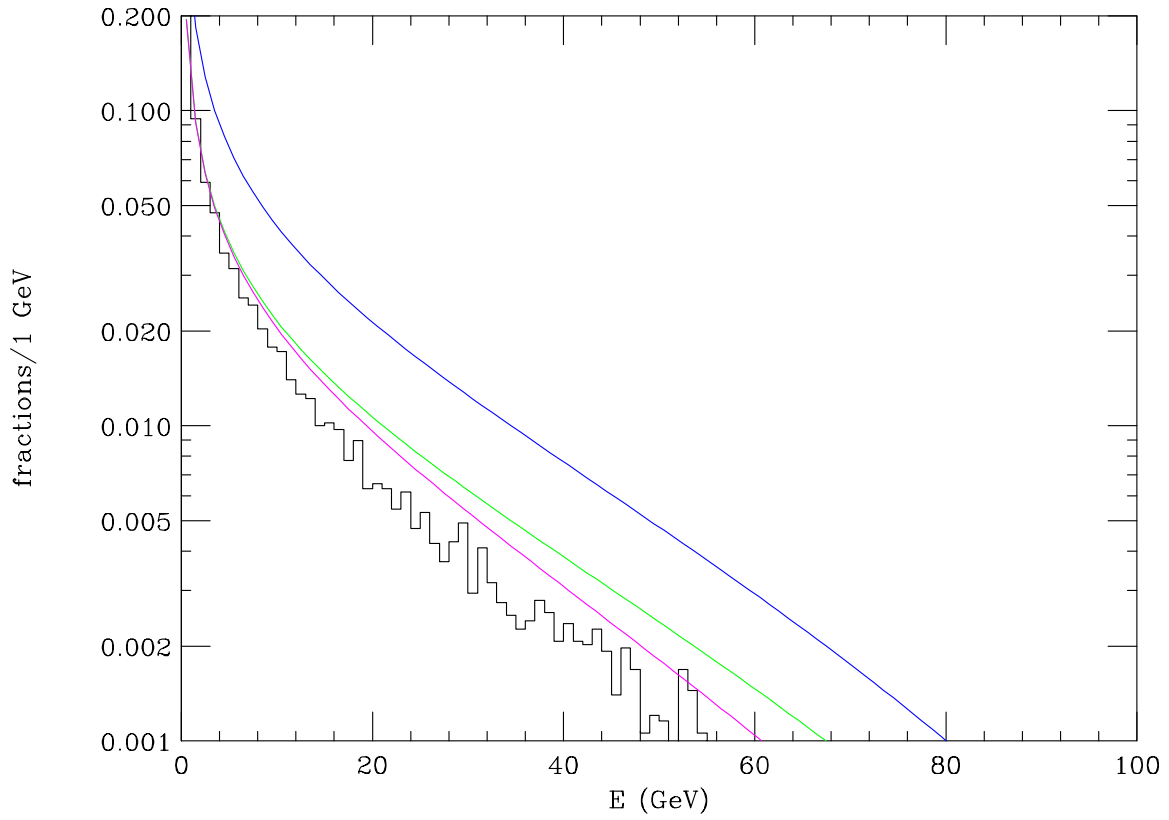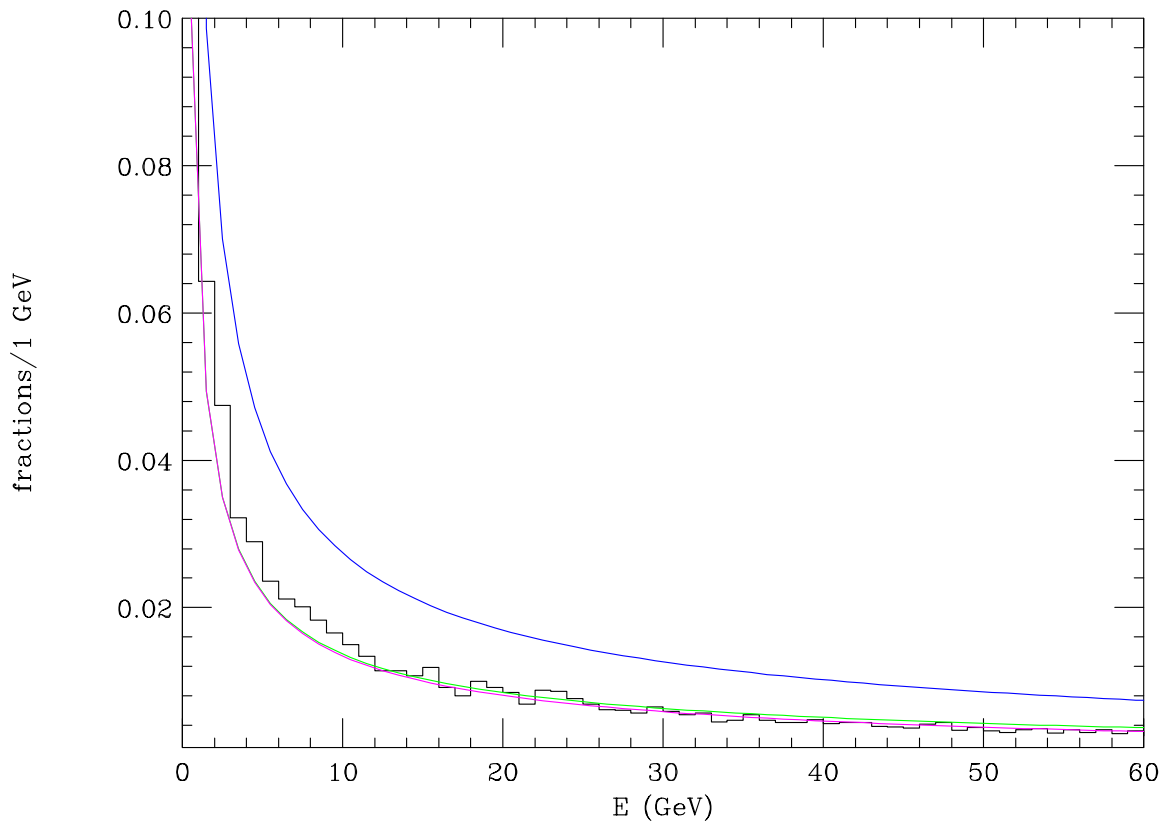Figure 3: Blowup of Figure 1 concentrating on the highest-x bins.

Figure 4: Comparison of Guinea Pig simulation data on the electron energy spectrum with various analytic approximations for the NLC-1000B design parameters: blue-P1, green-P2, red-YC1 over YC0, magenta-C2.

Figure 5: Figure 4 with a logarithmic scale, showing the low-energy, low-x tail of the distribution.

Figure 6: Blowup of Figure 4 concentrating on the highest-x bins.

Figure 7: Comparison of Guinea Pig simulation data on the photon energy spectrum with various analytic approximations for the NLC-500B design parameters: blue-PG1, green-PG2, magenta-CP2.
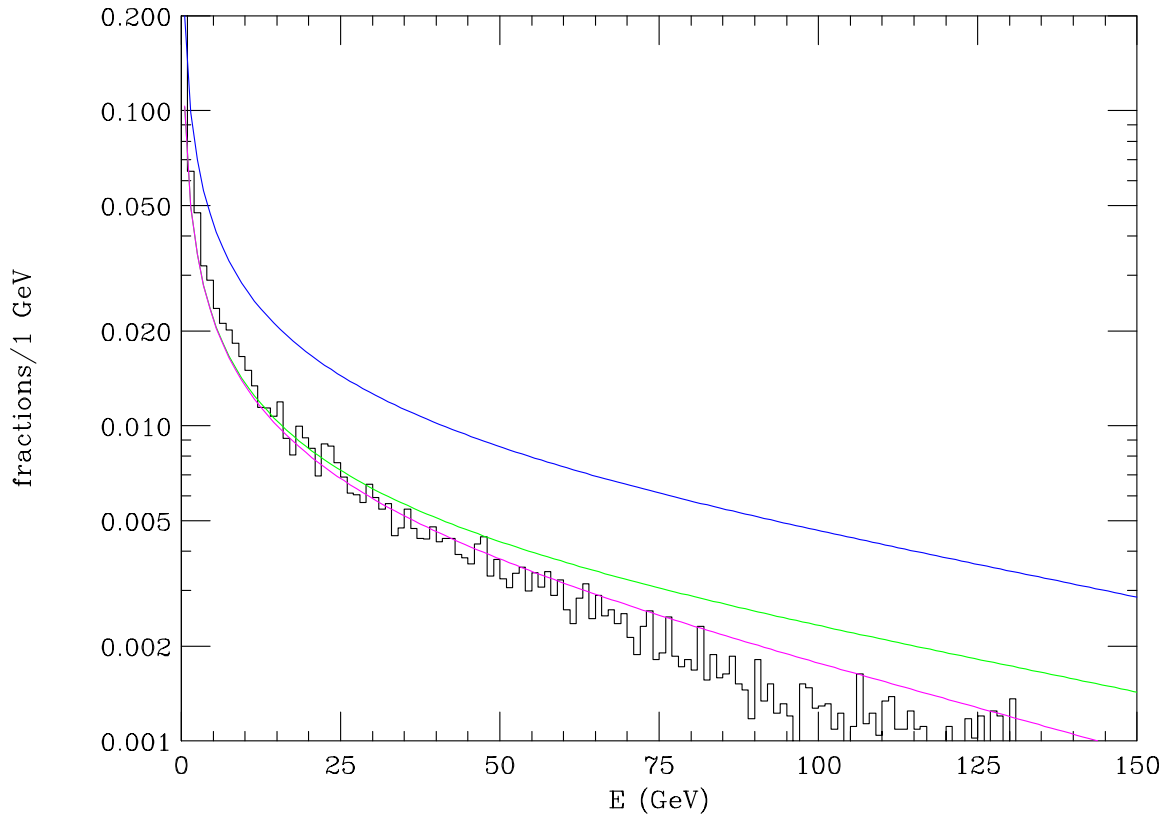
Figure 8: Figure 7 with a logarithmic scale, showing the high-energy, tail of the distribution.

Figure 9: Comparison of Guinea Pig simulation data on the photon energy spectrum with various analytic approximations for the NLC-1000B design parameters: blue-PG1, green-PG2, magenta-CP2.

Figure 10: Figure 9 with a logarithmic scale, showing the high-energy, tail of the distribution.
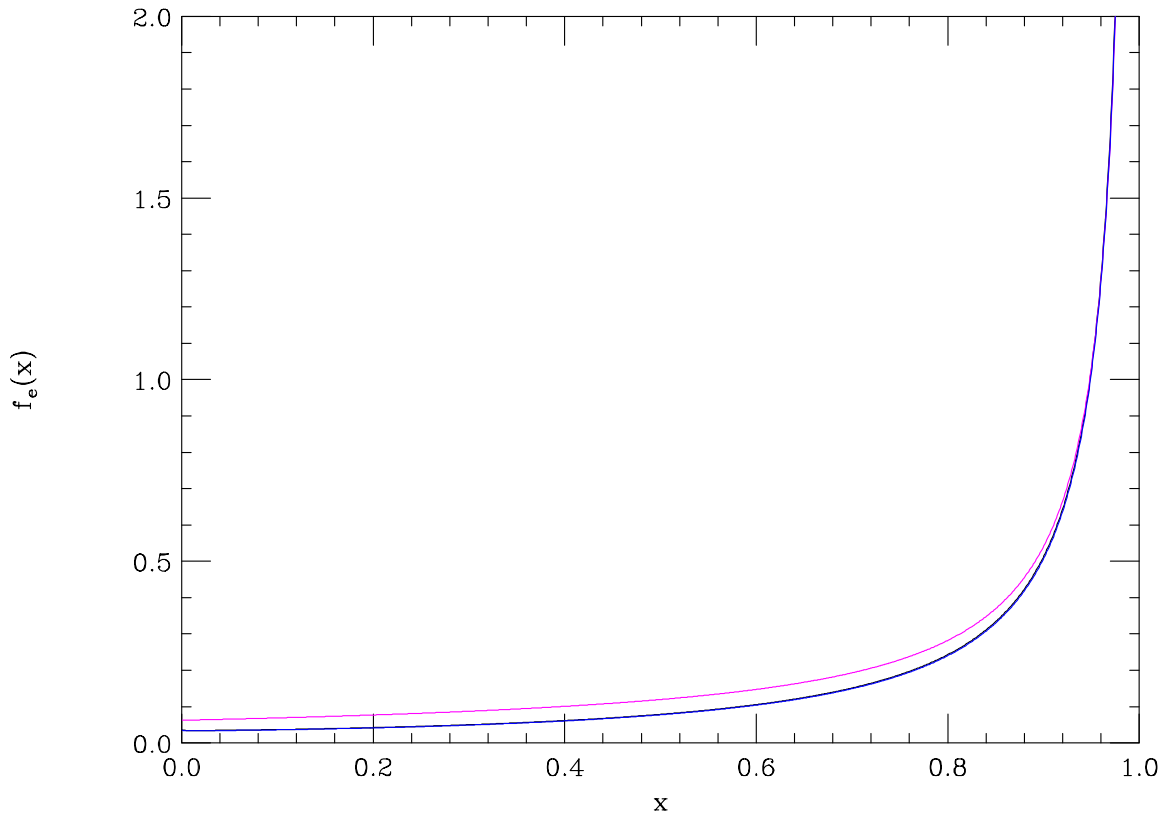
Figure 11: Comparison of various approximations to the electron spectra resulting from initial-state radiation: black-KF, blue-SJ, magneta-first line of (22).
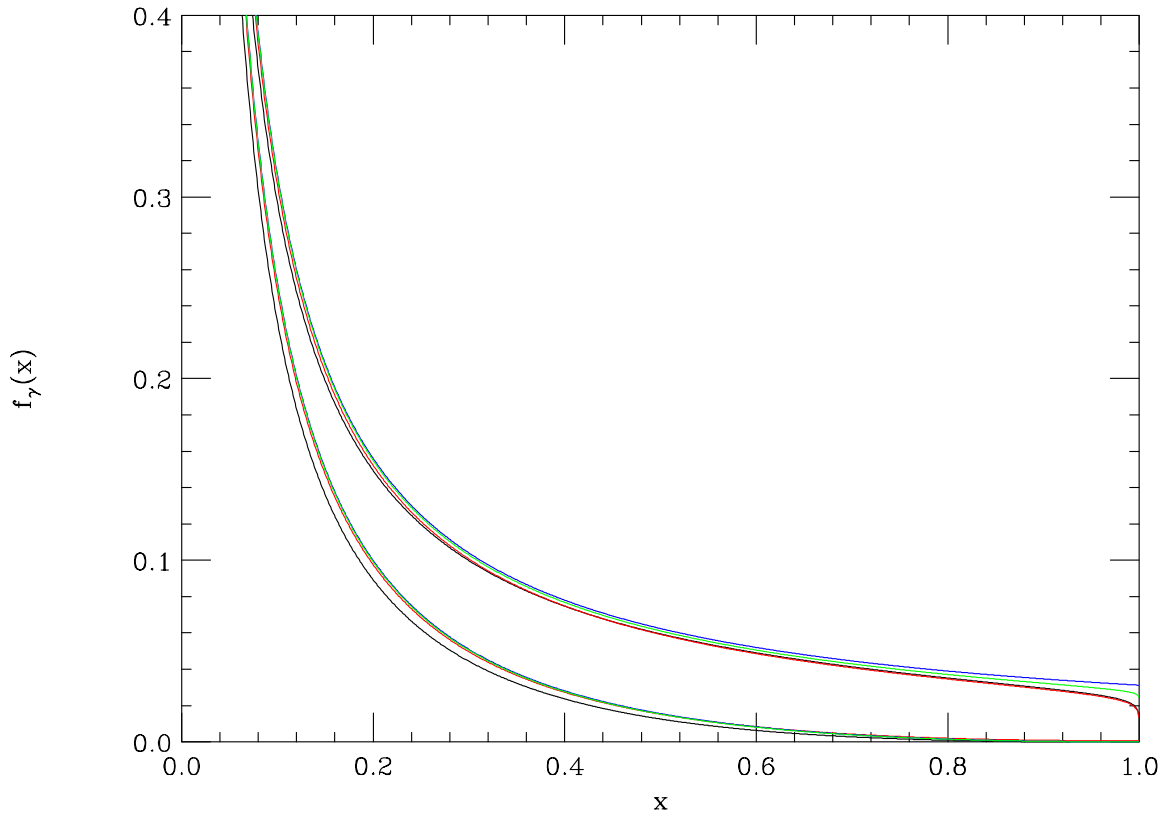
Figure 12: Comparison of various approximations to the polarized photon spectra of initial-state radiation: black-BKT, red-improved BKT, blue-simple GL, green-improved GL.
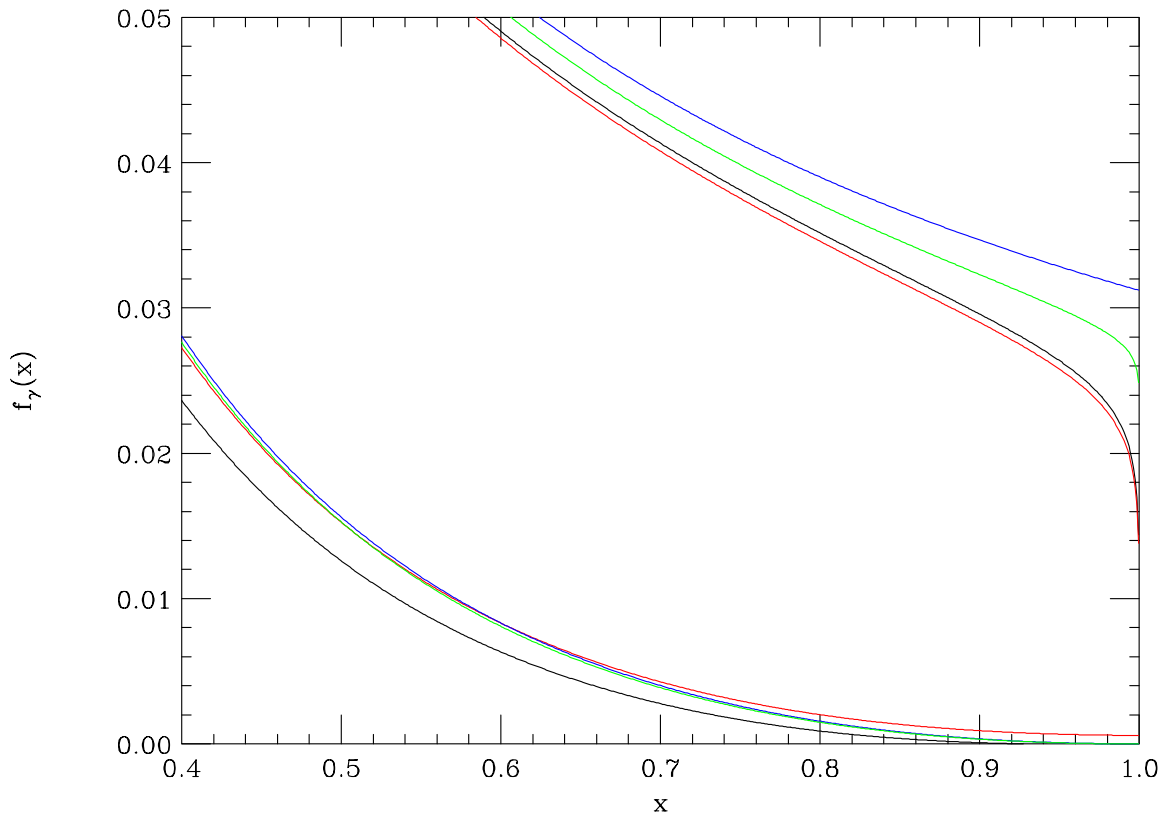
Figure 13: Blowup of Figure 12 showing the behavior of the various approximations near the endpoint.