SINGLE AND MULTIPLE TOUSCHEK EFFECTS

J. Le Duff
Laboratoire de l'Accélérateur Linéaire, Orsay, France

1. INTRODUCTION

Particles in a circulating bunch execute transverse betatron oscillations around the equilibrium orbit. Since the transverse velocities are statistically distributed, these particles can be scattered by collisions so transferring transverse momenta into longitudinal momenta. Such a change in the direction of the particle's momentum can lead to a strong variation of its energy, due to a relativistic effect: if the relative energy variation exceeds the energy acceptance of the machine the particle is lost.

The effect was first recognized by B. Touschek (1963) on the small e+e− storage ring ADA, by observation of the stored bunch lifetime. The theoretical model was first worked out by C. Bernadini, B. Touschek and J. Haissinski [1-4] and later on by some others [5,6].

For low and medium energy electron storage rings, operating with flat, dense beams, only the horizontal betatron motion produces sufficiently high relative transverse velocities for the energy variations to lead to particle loss. The theory, however, did not predict accurately experimental lifetime observations in the lowest part of the ADA energy range where measured values were much better than those calculated. H. Bruck first pointed out that this could be due to an anomalous increase in the transverse beam dimensions. A model was worked out [7,8] showing that smaller energy transfers, not contributing directly to particle losses, were responsible for the increase. These small transfers, having a high probability, act like an additional noise on the quantum radiation and, as a consequence, the balance from the damping of betatron oscillations leads to a new steady state.

Both the single and multiple Touschek effects appear in H. Bruck's book [9]. The following presentation follows essentially the original work with, however, some extension to the case of strong focusing machines. In 1974 the multiple effect theory was generalized by A. Piwinski [10] to take proton storage rings into account and was renamed 'Intrabeam Scattering' [11,12].

For a long time the Touschek effects were almost forgotten since the operating conditions of e+e− circular colliders, with large emittances to minimize beam-beam effects, made them negligible. More recently, however, they came back into evidence with the construction and operation of very small emittance electron (or positron) storage rings dedicated to synchrotron radiation physics.

2. SINGLE TOUSCHEK EFFECT

2.1 A rough estimate of the effect

Particles in a bunch execute transverse betatron oscillations around the equilibrium orbit (Fig. 1a). In the moving frame of the bunch the motion becomes purely transverse, neglecting the slow synchrotron motion. Coulomb scattering will occur for particles having different transverse velocities and will result in an energy transfer from the transverse plane to the longitudinal direction (Fig. 1b).

A particle with horizontal betatron amplitude \( \dot{x} \) will have a maximum horizontal velocity
Fig. 1  Coulomb scattering of two particles inside the bunch. (a) Laboratory system, (b) Centre-of-mass system.

\[ x' = \frac{\hat{x}}{\beta_x} = \frac{\hat{x}}{\lambda_x} = \frac{p_x}{p} \]

where \( \lambda_x \) is the betatron wavelength and \( \beta_x \) the envelope function. It corresponds to a transverse momentum \( p_x = p \hat{x} / \beta_x \). Consider a typical machine with \( \hat{x} = 1.10^{-4} \) m and \( \beta_x = 10 \) m with an operating energy of 800 MeV. The transverse momentum can be as much as 8 keV. When transferred into the longitudinal direction it becomes \( \Delta E = \gamma p_x = 12.5 \) MeV or \( \Delta E / E = 1.56\% \) which obviously is of the order of magnitude of a typical energy acceptance (e.g. RF acceptance). Since the particles have a Gaussian transverse energy distribution, events with higher energy transfers can occur.

Generally, storage rings operate with flat beams such that the vertical size is at least an order of magnitude smaller than the horizontal one. Hence the vertical betatron motion will contribute much less to the losses.

2.2 Total cross section for particle losses

It is usual to consider the Coulomb scattering of two particles in their centre-of-mass (c.m.) system in which the particles will have equal but opposite momenta. After the collision each particle is scattered by an angle \( \theta \) as shown in Fig. 2 for one particle. The number of particles \( dN \) which are scattered into the differential solid angle \( d\Omega \) is generally represented by the formula

\[ dN = d\sigma \ N \ n \ dx \]

where \( N \) is the number of test particles with velocity \( v = dx/dt \) hitting the target of density \( n \), and where \( d\sigma \) is the differential cross section for the physical process: \( d\sigma = \sigma d\Omega \). Integration over all values of interest for the solid angle would give the corresponding total cross section \( \sigma_T \). The rate of events for such a physical process is then

\[ \frac{dN}{dt} = \sigma_T \ v \ N \ n \]

where \( N = n \ dV \) since the test beam and the target are common.

In the present case, the Coulomb scattering differential cross section for electrons (or positrons) is given by the Möller formula (non-relativistic case)
Fig. 2 Coulomb scattering in the c.m. system

\[
\frac{d\sigma}{d\Omega} = \frac{4r_0^2}{(v/c)^4} \left[ \frac{4}{\sin^4 \theta} - \frac{3}{\sin^2 \theta} \right]
\]

where \(v\) is the relative velocity in the c.m. system.

The momentum transfer into the longitudinal direction is

\[\Delta p = p_x|\cos \chi|,\]

\(2p_x\) being the relative transverse momentum. The particle is lost if \(\gamma \Delta p \geq \varepsilon_{RF}\) where \(\varepsilon_{RF}\) is the momentum acceptance of the RF (assuming this is the limiting acceptance). Hence the total cross section for particle losses is obtained by integrating the differential cross section over angles such that:

\[|\cos \chi| \geq \varepsilon_{RF} / \gamma p_x.\]

Putting \(\mu = \varepsilon_{RF} / \gamma p_x\) one obtains

\[\sigma_T = \int_{|\cos \chi| \geq \mu} d\sigma\]

and since \(\cos \theta = \sin \chi \cos \phi\)

\[\sigma_T = \frac{4r_0^2}{(v/c)^4} \int_0^{\pi} \sin \chi \, d\chi \cdot 2 \int_0^{\pi} d\phi \left[ \frac{4}{(1-\sin^2 \chi \cos^2 \phi)^2} - \frac{3}{(1-\sin^2 \chi \cos^2 \phi)} \right].\]

The first integral is bounded to zero and \(\arccos \mu\) knowing that the final state is the same for the two particles. Performing the integration leads to:

\[\sigma_T = \frac{8\pi r_0^2}{(v/c)^4} \left[ \frac{1}{\mu^2} - \frac{1 + \ln \mu}{\mu^2} \right]\]

remembering that this formula is valid in the c.m. system.
2.3 Touschek lifetime

The total rate of collisions leading to losses can be written as

\[
\frac{dN}{dt} = \int_V \frac{\sigma_T}{\gamma^2} \nu n^2 \, dV
\]

where \( V \) is the bunch volume. The factor \( \gamma^2 \) takes account of the Lorentz transformation of the product \( \sigma_T \nu \) from the c.m. to the laboratory system.

Since we are considering only effects which take place in the horizontal plane the integration is automatically performed in the vertical and longitudinal planes, hence:

\[
\frac{dN}{dt} = \frac{N^2}{\gamma^2} \frac{1}{4 \pi \sigma_\parallel \sigma_\perp} \int_V \sigma_T \nu \rho(x_1, x_1') \rho(x_2, x_2') \, dV
\]

where

\[
\rho(x, x') = \frac{\beta_x}{2 \pi \sigma_x^2} e^{-\left[ x^2 + \left( \beta_x x' - 1/2 \beta_x x \right)^2 \right]/2 \sigma_x^2}
\]

is the horizontal phase space distribution function in a strong focusing machine, and where \( x_1 = x_2 \) in order for the collision to take place. Note that the contribution to the horizontal transverse velocity from the dispersion function has been neglected.

In the laboratory system the relative velocity is simply:

\[
\left( \frac{\nu}{c} \right)_{\text{lab}} = \dot{x}_2 - \dot{x}_1 .
\]

The integral now becomes

\[
\frac{dN}{dt} = \frac{N^2}{\gamma^2} \frac{\beta_x^2}{16 \pi^3} \frac{\sigma_\parallel^2 \sigma_\perp}{\sigma_x^4} \int_{-\infty}^{+\infty} \sigma_T(v) \nu e^{-\left( A \beta_x^2 + B \beta_x + C \right)/2 \sigma_x^2} \, dx_1 \, dx_1' \, dx_2
\]

where

\[
A = 2 + \frac{1}{2} \beta_x^2, \quad B = -\beta_x \beta_x' (x_1' + x_2') \quad \text{and} \quad C = \beta_x^2 \left( x_1'^2 + x_2'^2 \right) .
\]

Since \( \sigma_T \) and \( \nu \) do not depend on the position \( x_1 \) the integration with respect to that variable is straightforward

\[
\frac{dN}{dt} = \frac{N^2}{\gamma^2} \frac{\beta_x^2 \sqrt{2 \pi}}{16 \pi^3} \frac{\sigma_\parallel^2 \sigma_\perp}{\sigma_x^4} \int_{-\infty}^{+\infty} \sigma_T(v) \nu e^{\frac{\beta_x^2 \beta_x'^2 (x_1' + x_2')^2}{8 A \sigma_x^2} - \frac{\beta_x^2}{2 \sigma_x^2} \left( x_1'^2 + x_2'^2 \right)} \, dx_1' \, dx_2' .
\]
If we now make the following change to the variables:

\[ u_1 = x_2, \]
\[ u_2 = x_2 - x_1, \]

such that \( \sigma_T(v)v \) will only depend on \( u_2 \) and the integration on \( u_1 \) is simplified, then

\[
\frac{dN}{dt} = N^2 \frac{\beta_x}{\gamma^2} \frac{1}{16\pi^2 \sigma_z^2 \sigma_s} \int \sigma_T(v) v e^{-\frac{\beta^2}{2\sigma^2} u_2^2} du_2.
\]

Note that \( u_2 = \left( v / c \right)_{\text{lab}} \); hence one has

\[ u_2 = \gamma^{-1}v/c \]

where \( v \) is the relative velocity in the c.m. system. One can then write

\[
\left( \frac{dN}{dt} \right)_{\text{loss}} = 2 \left( \frac{dN}{dt} \right)_{\text{collision}} = 2 N^2 \frac{\beta_x}{\gamma^2} \frac{1}{16\pi^2 \sigma_z^2 \sigma_s} \int \sigma_T(v) v e^{-\frac{\beta^2}{4\sigma^2} (v)^2} \frac{1}{\gamma} d\left( \frac{v}{c} \right)
\]

But we can also introduce the individual particle momentum \( p_x \) in the c.m.

\[ p_x = \frac{\beta}{2} \frac{m_0 v}{2} \]

where \( m_0 \) is the rest mass. The loss rate then becomes

\[
\left( \frac{dN}{dt} \right)_{\text{loss}} = N^2 \frac{\beta_x}{\gamma^2} \frac{1}{8\pi^2 \sigma_x^2 \sigma_z \sigma_s} \int_{\epsilon_{RF}/\gamma} \sigma_T(p_x) \frac{2p_x}{m_0} e^{-\frac{p_x^2}{\sigma_x^2 \gamma m_0 c^2}} \left[ \frac{1}{p_x^2} \frac{\epsilon_{RF}^2}{\epsilon_{RF}^2 - 1 + \ln \left( \frac{\epsilon_{RF}}{\gamma P_x} \right)} \right] \frac{2}{m_0 c} dp_x
\]

where the first bound means that particles can be lost only if their initial momentum exceeds the momentum acceptance in the c.m.

Using the value \( \sigma_T(v)v \) as calculated above and introducing the quantity:

\[ \sigma_p = \frac{\gamma m_0 c \sigma_x}{\beta_x} \]

which is known to be the r.m.s. of individual transverse particle momenta, one obtains

\[
\left( \frac{dN}{dt} \right)_{\text{loss}} = N^2 \frac{\beta_x \gamma m_0 c^3}{\gamma^3} \frac{1}{4\pi \sigma_x \sigma_z \sigma_s} \int_{\epsilon_{RF}/\gamma} \frac{1}{p_x^2} \left[ \frac{\epsilon_{RF}^2}{\epsilon_{RF}^2 - 1 + \ln \left( \frac{\epsilon_{RF}}{\gamma P_x} \right)} \right] e^{-p_x^2/\sigma_p^2} dp_x
\]
which can be written in the following form, well known in the literature

\[
\frac{1}{N} \frac{dN}{dt} = \frac{1}{\tau} = \frac{N r_0^2 c}{8 \pi \sigma_s \sigma_s \gamma^2} D(\xi)
\]

where \( \tau \) is the lifetime and with the parameter definition:

\[
\lambda^{-1} = (\Delta E / E)_RF = \epsilon_{RF} / \gamma m_0 c
\]

\[
\xi = (\epsilon_{RF} / \gamma \sigma_p)^2
\]

and \( D(\xi) \) being the universal function

\[
D(\xi) = \sqrt{\xi} \left\{ \frac{3}{2} e^{-\xi} + \frac{\xi}{2} \int_{\xi}^{\infty} \ln \frac{u}{e^{-u}} du + \frac{1}{2} (3 \xi - \xi \ln \xi + 2) \int_{\xi}^{\infty} \frac{e^{-u}}{u} du \right\}
\]

for which the plot is shown in Fig. 3. Note that this function presents a flat maximum equal to 0.3 in the \( \xi \) range generally of interest.

![Plot of D(\xi)](image)

Fig. 3 Plot of \( D(\xi) \)

The formula shows a dependence of the parameters with the azimuth \( s \). This is true for the bunch dimensions but it can also be true for \( \lambda \) if the vacuum chamber constitutes the real energy acceptance (this can happen for instance when energy transfer occurs in a region of high dispersion). The average effect over the whole circumference is

\[
\frac{1}{\tau} = \frac{1}{2 \pi R} \int \frac{1}{\tau(s)} ds = \left\langle \frac{1}{\tau(s)} \right\rangle.
\]

Up to now we have neglected the influence of the dispersion function on the parameters of the Coulomb scattering. A more precise study would include the assumption that the overall phase space coordinates of the two particles are:

- particle 1 \( [x_1, x_1', D_x \epsilon_1, D_x \epsilon_1'] \)
- particle 2 \( [x_2, x_2', D_x \epsilon_2, D_x \epsilon_2'] \)
where \( \varepsilon = \Delta E / E \) is the relative energy deviation, and \( D_x \) the horizontal dispersion function.

Two particles having the same transverse position are now defined as

\[
x_2 = x_1 + D_x (\varepsilon_1 - \varepsilon_2)
\]

and the relative velocity in the laboratory system becomes

\[
\left( \frac{\gamma}{c} \right)_{\text{lab}} = (x_2' + D_x' \varepsilon_2) - (x_1' + D_x' \varepsilon_1).
\]

All the integrations performed under these more general assumptions, lead to the formula:

\[
\frac{1}{\tau} = \frac{N r_0^2 c}{8 \pi \sigma_x \sigma_y \sigma_{x\beta} \sqrt{1 + \sigma_{xs}^2 / \sigma_{x\beta}^2}} \frac{\lambda^3}{\gamma} D_x(\xi)
\]

where \( \sigma_{x\beta} \) and \( \sigma_{xs} \) are respectively the r.m.s. horizontal contributions from betatron and synchrotron motions

\[
\sigma_x^2 = \sigma_{x\beta}^2 + \sigma_{xs}^2.
\]

The other parameters have the same meaning as before, but with the more general formula for the r.m.s. relative momentum distribution

\[
\sigma_p = \frac{\gamma m_0 c}{2 \sqrt{C_1}}
\]

with:

\[
C_1 = \frac{\beta_x^2}{4 \sigma_{x\beta}^2} - \frac{B_1^2}{4 A_1}
\]

\[
A_1 = \frac{1}{4 \sigma_x^2} + \frac{1}{4 \sigma_{x\beta}^2} \left[ D_x^2 + \left( \beta_x D_x - \beta_x' D_x / 2 \right)^2 \right]
\]

\[
B_1 = \frac{\beta_x}{2 \sigma_{x\beta}} \left[ \beta_x D_x' - \beta_x' D_x / 2 \right]
\]

\[
\sigma_{xs}^2 = D_x^2 \sigma_x^2
\]

where \( \sigma_e \) is the r.m.s. of the relative energy spread.

The derivatives of the optical functions only appear in the argument \( \xi \). Since the function \( D(\xi) \) has a flat maximum they will not contribute very much to the lifetime. Finally the correction from the dispersion can only increase the lifetime, at least if the vacuum chamber is not the momentum aperture.

3. **MULTIPLE TOUSCHEK EFFECT**

3.1 **Basic idea**

Let us now examine the smaller scattering angles which are insufficient to produce particle losses but which nevertheless disturb the particle statistical distribution (noise source).
A first obvious consequence of the Coulomb collisions is a statistical change of the particle energies in the longitudinal phase space, leading to an increase of the energy spread.

A spontaneous energy deviation induces an energy oscillation around the nominal energy (synchrotron motion) damped by the synchrotron radiation. It is well known that a set of harmonic oscillators excited by random fluctuations, with r.m.s. $\langle \delta E^2 \rangle$ at the rate $\dot{N}$ will have a Gaussian distribution with standard deviation

$$\sigma_E^2 = \frac{1}{4} \dot{N} \langle \delta E^2 \rangle \tau_E$$

where $\tau_E$ is the damping time for the synchrotron motion.

A rapid change in the particle energy will give, in addition, a jump $\delta x$ of the radial particle position ($\delta x = D_x \delta E/E$), according to the finite dispersion function $D_x$. This leads to an induced betatron motion which is then also damped by the synchrotron radiation. The statistical result is a new steady state for the particle distribution in the horizontal plane of motion

$$\sigma_x^2 = \frac{1}{4} \dot{N} \langle \delta x^2 \rangle \tau_x$$

These r.m.s. values from multiple Coulomb scattering must be added quadratically to r.m.s. values related to other random effects (such as the quantum radiation) as soon as the noise sources appear to be independent

$$\sigma_{\text{total}}^2 = \sum \sigma_{\text{partial}}^2$$

### 3.2 Small-angle Coulomb scattering

At small angles the Möller formula for the differential cross section reduces to

$$\frac{d\sigma}{d\Omega} = \frac{16r_0^2}{(v/c)^4 \sin^4 \theta}$$

where $v$ is the relative velocity in the c.m. system that we will assume to be essentially horizontal. The angular change of the momentum gives a momentum component perpendicular to the horizontal axis (Fig. 2)

$$p_\perp = p_x \sin \theta \quad dp_\perp = p_x \cos \theta d\theta = -p_x d\theta$$

with

$$p_x = \frac{m_0 v}{2}$$

where $m_0$ is the rest mass of the electron since we still assume non-relativistic transverse motions.

Hence

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2 m_0^4 c^4}{p_\perp^4}$$
and since
\[ dΩ = -2\pi \sin θ dθ \]
\[ dσ = 2\pi \left( \frac{r_0 m_0^2 c^2}{p_x} \right)^2 \frac{1}{p_1^2} dp_1 \, . \]

This shows that the probability is higher for small perpendicular momentum transfers, and that the noise will mostly come from these small angle collisions.

Note that
\[ p_1 = \frac{2r_0 p_0^2}{b \, q} \]

where \( b \) is the impact parameter, \( p_0 = m_0 c \), and \( q = 2p_x \) the relative momentum.

3.3 Energy spread due to multiple scattering

The quantity of interest is, in the laboratory system:
\[ \dot{N} \langle \delta E^2 \rangle = c^2 \dot{N} \langle \delta p_x^2 \rangle \]

which, from the Lorentz transformation is equivalent to:
\[ \dot{N} \langle \delta E^2 \rangle = \frac{1}{2} c^2 \gamma \dot{N} \langle \delta p_1^2 \rangle_{c.m.} \]

where the factor 1/2 takes account of the fact that the probability is the same for transfers occurring in the vertical and longitudinal directions.

Considering one test particle, the rate of events from multiple scattering with the rest of the bunch is
\[ \dot{N} = σ \, ν \, n \]

and since
\[ n_{c.m.} = γ^{-1} n_{lab} \]

one can write
\[ \dot{N} \langle \delta E^2 \rangle = \frac{1}{2} c^2 n \{ σ \, ν \langle \delta p_1^2 \rangle \}_{c.m.} \, . \]

The mean quadratic transferred momentum can be obtained from the differential cross section:
\[ \langle \delta p_1^2 \rangle = \frac{\int p_1^2 dσ}{\int dσ} = \frac{1}{σ} \int p_1^2 dσ \]

where the integral is performed over all transfers corresponding to a given relative velocity in the c.m.
\[ \sigma_0^2 \Delta \approx 2 \pi r_0^2 m_0^2 c^2 \leq \frac{1}{p_x} \ln \left( \frac{p_{\perp \max}}{p_{\perp \min}} \right) \]

and where \[ p_{\perp \max} = p_x . \]

The minimum momentum transfer is related to the maximum distance which can occur between two particles. In the original model this was taken as the beam half-height. However, at long distances, the interaction between two particles is disturbed by the presence of the other particles (collective effects). Hence a more realistic maximum distance, or maximum impact parameter for free binary collisions would be:

\[ b_{\max} \equiv r^{-1/3} \]

and the corresponding lower limit for the momentum transfer becomes

\[ p_{\perp \min} = \frac{2 r_0 p_0^2}{b_{\max} q} . \]

Hence one can write

\[ \sigma_0^2 \Delta \approx 2 \pi r_0^2 m_0^2 c^2 \leq \frac{1}{p_x} \ln \left( \frac{p_x}{p_m} \right) \]

with

\[ \left( \frac{p_x}{p_m} \right)^2 = \frac{p_x^2 b_{\max}}{r_0 p_0^2} \]

where

\[ p_m = p_0 \sqrt{\frac{r_0}{b_{\max}}} \]

can be considered as a lower limit on the effective initial momentum (this effect comes from the use of the classical treatment).

Now, in order to take into account all possible relative velocities (or momenta) and all possible particle locations within the bunch, one needs to average the expression:

\[ n \{ \sigma_0 \sigma_0^2 \} \}_{c.m.} . \]

However, considering Gaussian distributions and the fact that velocity distribution is independent of particle location one has

\[ \bar{n} \{ \sigma_0 \sigma_0^2 \} \}_{c.m.} = \bar{n} \{ \sigma_0 \sigma_0^2 \} \}_{c.m.} . \]

with

\[ \bar{n} = N \frac{\iiint \rho \cdot \dot{p} \, dV}{\iiint \rho \, dV} = \frac{N}{2^3 \pi^{3/2} \sigma_x \sigma_z \sigma_s} \]
and

\[
\left\{ \sigma \langle \delta p^2 \rangle \right\}_{c.m.} = \int \frac{2p_x}{m_0} \sigma (\delta p^2) P(p_x) dp_x
\]

where \( P(p_x) dp_x \) is the differential probability acting on the variable \( p_x \).

Since individual particle momenta have a Gaussian distribution, and since

\[
p_x = \frac{|\vec{p}_1 - \vec{p}_2|}{2}
\]

then

\[
\sigma^2_{p_x} = \frac{1}{2} \sigma^2_p.
\]

The transverse momentum being invariant in a Lorentz transformation

\[
\sigma_p = \gamma \ p_0 \sigma_x
\]

where \( \sigma_x \) is given from betatron dynamics considerations:

\[
\sigma_x = \frac{\sigma_x}{\beta}.
\]

Finally the bracket becomes

\[
\left\{ \sigma \langle \delta p^2 \rangle \right\}_{c.m.} = \frac{4\sqrt{\pi} r_0^2 m_0^3 c^4}{\sigma_p} \int_{p_m}^{\infty} \frac{1}{p_x} \ln \left[ \frac{p_x}{p_m} \right]^2 e^{-p_x^2/\sigma_p^2} dp_x.
\]

With a suitable change of variable

\[
\chi = \left[ \frac{p_x}{\sigma_p} \right]^2
\]

it takes the more convenient form:

\[
\left\{ \sigma \langle \delta p^2 \rangle \right\}_{c.m.} = \frac{2\sqrt{\pi} r_0^2 p_0^3 c}{\sigma_p} f(\chi_m)
\]

with

\[
f(\chi_m) = \int_{\chi_m}^{\infty} \frac{1}{\chi} \ln \left( \frac{\chi}{\chi_m} \right) e^{-\chi} d\chi
\]

\[
\chi_m = \left[ \frac{p_m}{\sigma_p} \right]^2 = \frac{r_0 p_0^2}{b_{\max} \sigma_p^2}
\]
Now putting all terms together one gets for the r.m.s. energy spread from multiple scattering:

\[
\sigma^2_E = \frac{N\sigma_0^2 P_0 C^3 \beta_x}{2^2 \pi \gamma \sigma_x^2 \sigma_y \sigma_z} \tau_E f(\chi_m).
\]

The function \(f(\chi_m)\) is tabulated on Fig. 4 for a practical range of \(\chi_m\) values.

![Graph](image)

**Fig. 4 Plot of the function \(f(\chi_m)\)**

In practice the total energy spread will be the result of the contributions from the present effect and the quantum radiation. These two effects are not correlated so they add quadratically

\[
(\sigma^2_E)_{\text{total}} = (\sigma^2_E)_{\text{rad.}} + (\sigma^2_E)_{\text{Tousch.}}.
\]

This formula is self consistent since the bunch volume itself depends on the total energy spread.

### 3.4 Bunch lengthening and emittance growth

In the longitudinal phase space, energy oscillations are correlated with phase oscillations. The equilibrium distribution of particles around the synchronous phase is described by the r.m.s. bunch length directly proportional to the r.m.s. energy spread:

\[
\sigma_s = R \alpha \frac{\omega_s}{\Omega_s} \frac{\sigma_E}{E_s}
\]

where \(R\) is the physical ring radius, \(\omega_s\) the angular revolution frequency, \(\Omega_s\) the angular synchrotron frequency, \(E_s\) the nominal energy, and \(\alpha\) the momentum compaction.

Since the multiple Touschek effect corresponds to random energy changes, the horizontal betatron motion and its equilibrium state will be affected due to finite dispersion

\[
\sigma_x = \left[\beta_x U_x + D_x^2\right]^{1/2} \frac{\sigma_E}{E}
\]

where the invariant \(U_x\) is given by the lattice optical functions:
\[ U_x = \frac{J_x}{J_z} \left( \frac{1}{\beta_x} \left[ D_x^2 + \left( \beta_x D_x - \frac{1}{2}\beta_x D_x \right)^2 \right] \right) . \]

Note that in this expression the averaging is made in the bending magnets only.

Generally, with no bending in the vertical plane, the r.m.s. vertical beam dimension comes from residual coupling and is expressed through a coupling coefficient \( K \):

\[ U_z = \frac{K}{\beta} U_x \]

\[ \sigma_z = \sqrt{\frac{K}{\beta} U_x \frac{\sigma_E}{E_s}} . \]

Hence the effective bunch volume will grow as \( \left( \frac{\sigma_E}{E_s} \right)^3 \) and \( \sigma_p \) as \( \left( \frac{\sigma_E}{E_s} \right)_{\text{total}} \).

Considering the \( f(\chi_m) \) is not a fast varying function of \( \chi_m \) the multiple Touschek effect will decrease approximately as

\[ \left( \frac{\sigma_E}{E_s} \right)_{\text{total}} \propto \frac{1}{\left( \frac{\sigma_E}{E_s} \right)^3} . \]

this behaviour remaining valid for all beam dimensions.

Since the natural bunch volume due to quantum radiation decreases when the energy decreases, and since the damping time increases rapidly at the same time, it is clear that the effect increases rapidly with decreasing ring energy. Hence a good starting approach consists of evaluating the effect in the absence of synchrotron radiation and then trying to locate the energy threshold for the multiple Touschek effect to occur (Fig. 5).

![Graph showing energy threshold for multiple Touschek effect](image)

**Fig. 5** Energy threshold for multiple Touschek effect

In that case the practical formulae for the effect become:

\[ \left( \frac{\sigma_E}{E_s} \right)^6 = \frac{N}{\epsilon_0 \beta_x \epsilon_x \Omega_x f(\chi_m)} \frac{r_0^2 \beta_x \tau_E \Omega_x f(\chi_m)}{2^5 \pi \gamma^3 \left[ \beta_x U_x + D_x^2 \right] \sqrt{K \beta_x U_x} \alpha} \]

\[ \chi_m = \frac{N^{1/3} r_0^{2/3} Q_{1/3}}{2 \sqrt{\pi \gamma^2 \left[ \beta_x U_x + D_x^2 \right]^{3/2}}} \left( \frac{K \beta_x U_x}{R} \right)^{1/6} \alpha^{1/3} \left( \frac{\sigma_E}{E_s} \right)^{-3} \]
where $Q_s = \Omega_s / \omega_s$.

The system is solved by putting

$$\left(\frac{\sigma_E}{E_s}\right)^6 = A f(\chi_m)$$
$$\left(\frac{\sigma_E}{E_s}\right)^3 = B \frac{1}{\chi_m}$$

where $\chi_m$ is the solution of the following equation

$$f(\chi_m) = \frac{1}{A} \left(\frac{B}{\chi_m}\right)^2.$$  

* * *

REFERENCES


