AMPLITUDE AND PHASE DEVIATION IN THE
NLC DETUNED STRUCTURE

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I. INTRODUCTION AND MOTIVATION

The 1.8m accelerating structure for the NLC consists of 204 individual cells and two
 coupling cells each differing in phase from their full cell nearest neighbor by \(2\pi/3\) radians.
In order to avoid the deleterious effects of beam breakup it is required to minimize the
trailing wake field left behind by electron beam bunch trains traversing the iris-loaded
slow-wave structure. This is effected by decohering the higher order wakes induced by
the driving charge and this is accomplished for the NLC structure with a characteristic
tapering in the cells to produce a Gaussian spread in the higher order mode frequencies.

However, in constructing the accelerator the dimensions of the individual cells vary in a
monotonic adiabatic fashion from one end of the accelerator to the other. Because the
variation of dimensions is adiabatic the deviation of the phase shift from its design value
has been assumed to be small. It has also been assumed that the input and output cavities
matched to uniform structures with dimensions corresponding to the cells on the input and
output ends, will continue to be matched for the tapered structure. In the following work
we investigate the extent to which these expectations are realised.

II. DESIGN OF NLC GAUSSIAN DETUNED STRUCTURE

Our analysis is based upon a study of cell sections, each consisting of an iris and two half
cells. The NLC cavity is formed from 205 of these cell sections. The detuned structure is
designed by choosing \(a\), the iris diameter, and \(t\), the iris thickness such that the modal
spectrum of the higher order modes is Gaussian in nature. All cell sections have a
common length, chosen to yield a phase shift of \(2\pi/3\) at the design frequency 11.424 GHz.
Armed with these dimensions we used a scattering matrix method to determine the
eigenphase of each of the 205 cell sections as a function of \(b\), the common diameter of the
half cell on either side of the iris. This involved calculating the generalized scattering
matrix via a mode matching procedure (retaining forty modes in all sections), converting
the S-Matrix to a generalized wave-amplitude-transfer, or T-matrix, and solving for the
eigenvalues of the multi-mode T-matrix.
In practice we found that sufficiently accurate results on the phase were obtained by restricting ourselves to the single-mode S-matrix and corresponding T-matrix. That is, in the calculation of the S-matrix both the full set of modes in the iris and the full set of modes in the half cells decaying evanescently away from the iris are retained, but the effect on the T-matrix of decaying higher order modes from adjacent iris’s is neglected. This approximation greatly simplifies the transformation from the S-matrix to the T-matrix, and it allows us to obtain a simple analytical relation between the eigenphase and the elements of the single-mode T-matrix.

The eigenphase of each of the 205 iris’s having been determined as a function of the cell section radius \(b\), a secant method of iteration was used to determine the value of \(b\) yielding the prescribed \(2\pi/3\) eigenphase. The resulting cavity radius is shown in Fig. 1. As a check on its accuracy we recompute the phase from the dimensions associated with Fig. 1 and evaluate its deviation from the prescribed phase. The maximum phase deviation from the designed values is approximately \(10^6\) percent. The same procedure carried out for the generalized T-matrix produces a very similar functional dependence of the cell section diameter on the iris dimensions for the prescribed phase. The difference in the cavity radius, calculated by the single mode and the multi-mode method is shown in figure 2 and this difference reaches a maximum of \(.001\) at the input end of the structure. For the evaluation of the reliability of the adiabatic assumption, the difference is sufficiently small to allow us to use the single mode T-matrix method.

If these cells sections were assembled without modification to form an accelerating structure, there would be a small discontinuity of cell radius at the center of each cell. This is avoided in practice by setting the cell radius equal to the average of the computed cell section radii. As a result the cell sections are no longer exactly symmetrical and the mean of the radii on either side of the iris is no longer precisely equal to the value appropriate to the iris dimensions. These deviations in the cell section dimensions may give rise to a shift in the phase of the individual cells and perhaps to a cumulative phase shift for the structure.

In order to study the properties of assemblies of such cell sections, we adopt the following S-matrix procedure. The two ends of the assembly are coupled to an entrance channel (or “1” channel) and an exit channel (or “2” channel). We take both of these channels to be strictly periodic structures with cell diameter equal to that of the cell section to which it is attached and iris dimensions chosen so that the eigenphase of each of its cell sections is \(2\pi/3\). Since left going and right going waves are well defined for these periodic structures, S-matrix elements relating incoming waves to outgoing waves can be unambiguously defined. The resultant S-matrix has all the usual properties and the usual relations to the T matrix. \(S_{11}\) then provides a measure of the mismatch caused by the tapering, and the phase of \(S_{21}\) provides a measure of the phase shift across the entire assembly as modified by the tapering. In the actual structure the role of these channels is played by input and output waveguides which connect to a coupling cavity which matches them to periodic structures having the dimensions of the input and output cell sections of the assembly respectively. When the intervening assembly is tapered, the deviation from
perfect match and ideal phase advance per section should be well modeled by our procedure.'

III. ANALYSIS OF A SINGLE CELL SECTION WITHIN THE NLC STRUCTURE

We illustrate the procedure described above for a single cell section. First we specify the cell radii \( b_1 \) and \( b_2 \) on the left and right hand sides of the iris respectively and calculate the \( T \) matrix for the resultant cell section. Next, retaining the iris thickness, we calculate the iris diameters required to provide \( 2\pi/3 \) eigenphases for symmetric cell sections with cell radii \( b_1 \) and \( b_2 \) and calculate their eigenvectors. This enables \( S_{11} \), the reflection coefficient, and \( S_{21} \), the transmission coefficient to be obtained:

\[
S_{21} = \left( c_1^+ . T . c_2^+ \right)^{-1}
\]

\[
S_{11} = \left( c_1^+ . T . c_2^+ \right) \left( c_1^+ . T . c_2^+ \right)^{-1}
\]

(3.1)

Here \( T \) is the wave-amplitude matrix of the cell, \( c_2^+ \) (\( c_1^+ \)) is the eigenvector for the right hand channel (left hand channel) for waves traveling to the right and \( c_1^- \) is the eigenvector for the left hand channel for waves traveling to the left.

The results of applying this procedure to a specific example are shown in figs. 3 and 4. We choose iris 100 and specialize to the case in which the average of the two radii is that shown in Fig. 1 for iris 100. For cells near the center of the structure this specialization represents the actual state of affairs reasonably accurately.

Curves of the absolute values of \( S_{11} \) and \( S_{21} \) are shown in Fig 3, plotted as a function of the percentage deviation of \( b_1 \) from the mean value. The percentage phase deviation from 120 degrees of \( S_{21} \) is shown in Fig. 4. The least square error fit used to obtain a curve through the phase and through the magnitude of the transmission coefficient is found to adequate with a quartic polynomial. It is apparent that the properties of a cell section are very sensitive to small deviations from symmetry. The actual accelerator structure, however, has an asymmetric change in the cavity radius of approximately .01 percent, and thus inspection of the curves leads us to expect little reflection and almost perfect transmission of the wave through this particular cell.

Also shown (Fig. 4) is the phase of the transmission coefficient, and this remains at 120 degrees with a deviation of less than a tenth of a degree.

\[\text{We note that a recursive field matching technique, employed by S. Heifets \\& S. Kheifets (SLAC-Pub-5907) to determine the field amplitude and various other properties of a tapered structure, effectively eliminates the iris variables, and the complexity of the problem is a function of the number of modes within a cell rather than the number of cells, and in this respect is similar to the method delineated herein. They also discover that the deviation from the adiabatic hypothesis is small.}\]
IV. PHASE DEVIATION THROUGH COMPLETE NLC STRUCTURE

The procedure outlined in the previous sections is applied to a sequence of structures based upon iris 204 and 205, then iris 203,204, and 205, and so forth until one arrives at an assembly consisting of the entire 205 cell sections. The output channel is a periodic structure based upon iris 205 and its associated b, and the input channel is similarly constructed with reference to the iris of the input cell section. The diameters of the cells between the iris’s are tapered following the previously defined averaging procedure. To determine the properties of an assembly the T in the expressions of Eq. (3.1) is replaced by a product of the T matrices for each of its cell sections, the eigenvector on the right is that associated with iris 205, and the eigenvectors on the left are replaced by the eigenvectors of its left hand channel.

The magnitudes of the reflection coefficient and the transmission coefficient of the sequence of structures are displayed in figure 5 and figure 6, respectively. In these plots the abscissa corresponds to the cell section number of one end of the assembly. The overall magnitude of the reflection is seen to be very small but appears to be largest where the taper is largest. There is no tendency for the reflection to grow as cell sections are added. This is to be expected because the reflections from each section do not add up in phase.

The phase of $S_{21}$ for the same sequence of assemblies is shown in figure 7. The overall phase deviation, from the low energy end of the structure to the high end is remarkably small, and is of the order of one hundredth of one degree. There is very little reflection of waves throughout the structure, and the largest reflection occurs at the region with the greatest mismatch, i.e. at the input and output to the accelerator. It is interesting to note that the reflection coefficient of the middle energy cells shows a five-fold increase in the reflection coefficient over that of the uncoupled cells as calculated in the previous section (contrast figure 5 and figure 3a in the neighborhood of $\delta$ of approximately .01 percent). This enhancement in the reflection coefficient of the uncoupled cells enables one to understand the overall reduction in the reflection coefficient as one proceeds down the structure; the overall transmission increases because waves within the cavities of the coupled structure add destructively at the expense of the wave reflected in the initial section.

V. FIELD PROFILE ALONG THE COMPLETE STRUCTURE

The accelerator designated by cell number n=1 in the preceding section of course corresponds to the complete structure. We calculate the field on the axis for this structure by taking the eigenvector of the final HEC (high energy cell) and back propagating the vector through each individual cell and taking the product of the sum of the waves traveling to the left and right of the cell with the mode function for each particular cell:

$$E_z = c_e \cdot \eta \cdot T \cdot c^*_z \cdot Y e_z$$  \hspace{1cm} (5.1)
Here $\mathbf{e}_z$ is the mode function of the axial electric field evaluated on the axis of the accelerator, $T$ is the cumulative wave-amplitude matrix evaluated up to the particular cell under consideration, $Y$ is the waveguide admittance of a particular cell, $\mathbf{c}_e$ is the row vector $(1, 1)$, $\eta$ is a Pauli matrix, and $\mathbf{c}_2^+$ is the eigenvector of the last cell. The field amplitude along the axis obtained in this way is presented in Fig. 8a. The variation in the amplitude of the field is dominated by the variation in the group velocity. However, there is a small ripple in the amplitude which arises as a consequence of the small standing wave amplitude generated by the small reflections associated with the tapering. This interpretation is supported by the periodicity of the ripple as illustrated in Figs. 8b and 8c.

The actual fluctuations in the total phase along the accelerator, which reaches a maximum of approximately one degree, is attributable to standing waves being present within the structure and is also indicated by the fluctuation in the amplitude of the field down the linac (fig. 8). The cumulative phase deviation of the axial electric field profile averaged with respect to a single axial period (i.e. three cells) is illustrated in fig. 9. The largest change in the cell parameters occurs in the input and output to the accelerator and this is supported by the curve of fig. 9 which reveals the maximum mean phase deviation to be approximately 0.27 degrees. Furthermore, the minimum gradient of the taper occurs in the middle of the structure and the curves indicate the minimum phase deviation to be no less than 0.01 degrees at cell one hundred.

The field profile viewed from the frame of reference of the accelerated beam, traveling at the velocity of light, is given by the product of the complex electric field amplitude with a phase factor describing the transit through the individual cells:

$$\mathbf{E}_{2b} = \text{Re}\{\mathbf{E}_z e^{j\mathbf{a}_c} e^{j\psi_0}\}$$

(5.2)

where $\psi_0$ corresponds to the initial phase of the field. Integrating the field along the length of the linac gives rise to a potential function which, for a single electron, corresponds to the energy gain from the electromagnetic field. The initial phase $\psi_0$ is varied in order to maximize the energy transference to the beam over the complete structure. The value which maximizes the potential function is retained and the resulting field profile is shown in Fig. 10. This field profile is representative of the energy gained by the electron beam in traversing each cell. Again we notice a small fluctuation (of the order of ±1.5%) superposed on to a smooth variation attributable to the variation in group velocity.

VI. CONCLUSIONS ON THE OVERALL PHASE AND FREQUENCY SHIFT

We conclude from the preceding analysis that the adiabatic assumption which underlies the design of the detuned structure is well justified. The overall phase shift which develops

\[2\text{ See Appendix.}\]
across the complete 206 cell NLC detuned structure is computed to be of the order of one
degree, due primarily to the presence of a small reflected wave. However, the mean phase
shift is of the order of .25 degrees and this shift can be transferred into a frequency
deviation by computing the filling time. The dispersion curves for over twenty cells
(obtained by utilizing our mode matching method) has been used to evaluate the
dependence of group velocity on cell number, and the resulting curve is displayed in Fig.
11 (where $v_g$ is in units of c). The filling time of the structure, obtained by integrating the
reciprocal of the group velocity over the complete length of the accelerator, is
approximately 105 ns. Hence, the frequency deviation is of the order of a few kHz. Also,
in the design, the cells remain well matched throughout as is readily verified by inspection
of the curves of the reflection and transmission coefficient. The field and the energy gain
show a small amplitude oscillation which results from the small reflections that occur
along the structure; the dominant variation arises from the variation in the group velocity
along the linac. Because these reflections do not add in phase, there is no progressive
build up of reflection as one proceeds from the output to the input.

As an overall measure of the degradation attributable to departure from adiabaticity we
compute the ratio of the potential function computed above to that obtained from
averaging the same quantity for the uniform structures formed from the individual cell
sections. We find this to be indicative of a degradation of .159 percent, which provides
strong justification for the conclusion that the adiabatic assumption is well justified.

VII. ACKNOWLEDGMENT

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Bane and J. Wang.
VIII. APPENDIX:
PROPERTIES OF PSEUDO-UNITARY AMPLITUDE TRANSFER MATRIX

The matrix which transfers the wave amplitudes on the right hand side (region 2) of a particular junction to the left hand side (region 1), the T matrix, or wave-amplitude-transfer matrix, is defined in terms of the amplitude vectors by:

\[
\begin{pmatrix}
    c_1' \\
    c_2'
\end{pmatrix}
= \begin{pmatrix}
    T_{11} & T_{12} \\
    T_{21} & T_{22}
\end{pmatrix}
\begin{pmatrix}
    c_1 \\
    c_2
\end{pmatrix}
\]  

(6.1)

Here the superscripts l and r refers to waves traveling to the left and right respectively. In terms of the unitary S matrix, defined by the waves scattered off a junction:

\[
\begin{pmatrix}
    c_1' \\
    c_2'
\end{pmatrix}
= \begin{pmatrix}
    S_{11} & S_{12} \\
    S_{21} & S_{22}
\end{pmatrix}
\begin{pmatrix}
    c_1 \\
    c_2
\end{pmatrix}
\]  

(6.2)

it is readily shown that:

\[
T = \begin{pmatrix}
    S_{21}^{-1} & -S_{21}^{-1}S_{22} \\
    S_{11}S_{21}^{-1} & S_{12} - S_{11}S_{21}^{-1}S_{22}
\end{pmatrix}
\]  

(6.3)

Now it is immediately evident that Det(T)=1 and the pseudo unitary matrix is completely determined by two elements since $T_{22} = T_{11}^*$ and $T_{21} = T_{12}^*$. Indeed, the inverse of the pseudo-unitary T matrix is of the form:

\[
T^{-1} = \eta T^\dagger \eta
\]  

(6.4)

Here $\eta$ is of the form of a Pauli spin matrix:

\[
\eta = \begin{pmatrix}
    1 & 0 \\
    0 & -1
\end{pmatrix}
\]  

(6.5)

Furthermore, as the T matrix is symplectic:

\[
\tilde{T}JT = J
\]  

(6.6)

for the matrix J defined by:

\[
J = \begin{pmatrix}
    0 & 1 \\
    -1 & 0
\end{pmatrix}
\]  

(6.7)

the inverse is also given in terms of J by:
\[ T^{-1} = -JTJ \]  

(6.8)

since \( J^2 = -1 \) (I is the unit matrix). This is in contrast to the unitary property of the mono-mode scattering matrix whose inverse is simply the transpose of the conjugate of itself.

Further, from these relations and the following transformation:

\[
S = \begin{pmatrix}
T_{11}^{-1} & T_{21} - T_{22} T_{11}^{-1} T_{12} \\
T_{12} & -T_{21}^{-1} T_{12}
\end{pmatrix}
\]

(6.9)

the unitary properties of the mono-mode scattering matrix can be delineated. The above equations relating the scattering matrix to the wave-amplitude matrix and vice-versa, viz., (6.3) and (6.9), are valid for the single mode scattering matrix and the generalized scattering matrix.

The eigenvalues of the above generalized T-matrix are of the form \( \exp(\pm i \Psi) \) for waves lying outside the band gap region of the dispersion curve of particular cell. In practice we obtain the phase by employing a mode matching procedure\(^3\) to evaluate the scattering matrix of a particular cell, convert to a generalized T-matrix and search for the eigenvalues of the wave-amplitude matrix. The eigenvalues are reciprocal to each other and are each of unit magnitude, representing the phase difference across a junction under consideration, i.e. they are, of course, complex conjugates of each other. For the accelerating mode in the NLC structure, which is monopole in character, it is sufficient to retain a single mode analysis of the eigenvalues:

\[
\cos \Psi = \frac{1}{2} \text{tr}\{T\} = \frac{1-\Delta}{2S_{21}}
\]

(6.10)

Here \( \text{tr} \) represents the trace of the matrix, \( \Delta \) represents the determinant of the one dimensional scattering matrix (the single mode S matrix is extracted from the multi-mode, or generalized scattering matrix evaluated for each of the 206 cells), and \( S_{21} \) is the transmission coefficient of a particular cell. For higher order modes, such as the dipole and quadrupole modes, it is necessary to retain all modes because in this instance coupling between TM and TE becomes significant, and it is not possible to obtain a closed form relation between the phase and the scattering matrix.

The eigenvectors of the T matrix possess the interesting property that:

\[
x_2 / y_2 = (y_1 / x_1)^* \quad (6.11)
\]

\(^3\)Refer to R.M. Jones, et al, SLAC PUB 6162 for further details.
Here the eigenvectors have individual components $x$ and $y$. Furthermore, for a symmetric matrix the eigenvectors are completely real and for a non-symmetric junction the imaginary component is related to the real via a single variable $d\phi$:

$$\text{Im}\{x, / y_1\} = \text{Re}\{x, / y_1\} \tan d\phi$$  \hspace{1cm} (6.12)

The variable $d\phi$ defines the phase difference in the off diagonal elements of the $T$- matrix:

$$T = j\csc \theta \begin{pmatrix} e^{-j\phi} & e^{-j\phi} \cos \theta \\ -e^{j\phi} \cos \theta & -e^{j\phi} \end{pmatrix}$$  \hspace{1cm} (6.13)

The variables $\phi$, $d\phi$ and $\theta$ are confined to the range:

$$\begin{cases} -\pi / 2 < \theta \leq \pi / 2 \\ -\pi / 2 < d\phi \leq \pi / 2 \\ -\pi < \phi \leq \pi \end{cases}$$  \hspace{1cm} (6.14)

The eigenvectors traveling to the right (left) of the junction are normalized in the form:

$$c^\dagger H c = \pm 1$$  \hspace{1cm} (6.15)

reflecting the principle of conservation of complex power.

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CAPTIONS:

Fig. 1: Cavity radius computed to achieve $2\pi/3$ phase shift via the single eigenvector method

Fig. 2: Difference, in inches, between radius computed via the single eigenvector method and the complete multi-mode eigenvector method.

Fig. 3a: Magnitude of the reflection coefficient, for parameters of 100th cell, as a function of delta, the change in the radius of the cavity, such that the summation of the radius of the left hand side cavity radius and the right hand side cavity radius remains constant (the irises surrounding the cell are also continuously varied in order to preserve the specified phase shift).

Fig. 3b: Magnitude of the transmission coefficient as a function of delta (as defined in fig 3a).

Fig 4: Phase deviation as a function of delta (as defined in fig 3a)

Fig 5: Magnitude of the reflection coefficient as one progresses down the NLC detuned accelerating structure (computed from back-propagating the eigenvectors and resolving into the left-propagating and right-propagating waves)

Fig. 6: Magnitude of the transmission coefficient as one progresses down a detuned accelerating structure (computed by the method as in fig 5)

Fig 7: Percentage phase deviation along detuned accelerating structure (computed from the phase achieved in fig 6)

Fig 8a: Overall amplitude of the electric field on the axis of the accelerator (evaluated by back-propagating through the high energy cells, via T-matrices, through to the low energy cells).

Fig 8b: Field profile though low energy cells (all other parameters as in fig 8a)

Fig 8c: Field profile though medium energy cells (all other parameters as in fig 8a)

Fig 8d: Field profile though high energy cells (all other parameters as in fig 8a)

Fig 9: Mean triplet phase deviation from the prescribed 120 degrees (computed from the phase of the field in fig. 8 and averaged with respect to a complete period, i.e. three cells).

Fig 10: Overall amplitude of the electric field with respect to the accelerated beam, on the axis of the accelerator (evaluated by back-propagating through the high energy cells, via
T-matrices, through to the low energy cells). The area enclosed by this curve corresponds to the energy gained by the beam.

Fig 11: Group velocity as a function of cell number along the NLC detuned structure (evaluated from the gradient of dispersion curves of each cell, calculated via the generalized mode matching method)
Figure 1

Figure 2
Abs Reflection Coefficient vs Delta (0.259036 \times Ise)

Figure 3a

Abs Transmission Coefficient vs Delta 
(1 - 0.03355 \times x^2 - 0.00056279 \times x^4 \times Ise)

Figure 3b
Phase Devratron vs Delta (-4.16*x^2 + 0.11*x^4)

Figure 4

Magnitude of Reflection Coefficient (Continuously Matched Sections)

Figure 5
Figure 6

Figure 7
Figure 8c

Figure 8d
Group Velocity vs Cell for NLC Detuned Structure

Figure 11