

Methods to estimate emittance dilutions due to misalignment of accelerating structures without tracking

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1. Numerical method

Divide a beam into small slices. Final transverse position and angle of m-th slice are

$$\begin{aligned} x_m(f) &= \sum_{\lambda} R_{12}(\lambda \rightarrow f) \Delta x'_m(\lambda) \\ x'_m(f) &= \sum_{\lambda} R_{22}(\lambda \rightarrow f) \Delta x'_m(\lambda) \end{aligned} \quad (1)$$

where $R_{12}(\lambda \rightarrow f)$ and $R_{22}(\lambda \rightarrow f)$ are the elements of transfer matrix from position λ to the end of the linac. λ expresses a piece of accelerating structures which is short enough so that the transfer matrix can be treated as constant in each piece. $\Delta x'_m(\lambda)$ is change of angle of m-th slice of the beam at λ due to kick by the wakefield :

$$\Delta x'_m(\lambda) = e \sum_k q_k W(\lambda, z_m - z_k) [x_k(\lambda) - a(\lambda)] L_{\lambda} / E(\lambda) \quad (2)$$

k is index for slices, q_k charge of k-th slice, $W(\lambda, z_m - z_k)$ dipole transverse wakefunction of λ at distance $z_m - z_k$, z_m relative longitudinal position of m-th slice, $x_k(\lambda)$ transverse position of k-th slice at λ , $a(\lambda)$ misalignment of structure λ , L_{λ} length of structure λ and $E(\lambda)$ beam energy at λ . Defining 'sum wake' as

$$S_m(\lambda) \equiv \sum_k q_k W(\lambda, z_m - z_k) \quad (3)$$

and assuming the beam oscillation is negligibly small compared with the misalignment, the change of angle becomes

$$\Delta x'_m(\lambda) \approx -e S_m(\lambda) a(\lambda) L_{\lambda} / E(\lambda) \quad (4)$$

Emittance dilution is approximately,

$$\Delta \varepsilon \approx \frac{1 + \alpha^2}{2\beta} \overline{x^2} + \alpha \overline{xx'} + \frac{\beta}{2} \overline{x'^2} \quad (5)$$

where

$$\begin{aligned} \overline{x^2} &= \sum_m q_m x_m^2 / \sum_m q_m - \bar{x}^2 \\ \overline{xx'} &= \sum_m q_m x_m x'_m / \sum_m q_m - \bar{x} \bar{x}' \\ \overline{x'^2} &= \sum_m q_m x'^2_m / \sum_m q_m - \bar{x}'^2 \\ \bar{x} &= \sum_m q_m x_m / \sum_m q_m \\ \bar{x}' &= \sum_m q_m x'_m / \sum_m q_m \end{aligned} \quad (6)$$

At the end of the linac,

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$$\begin{aligned}
\overline{x^2(f)} &\approx e^2 \sum_{\lambda} \sum_{\lambda'} R_{12}(\lambda \rightarrow f) R_{12}(\lambda' \rightarrow f) \\
&\quad \times a(\lambda) a(\lambda') L_{\lambda} L_{\lambda'} T_{\lambda\lambda'} / E(\lambda) E(\lambda') \\
\overline{xx'(f)} &\approx e^2 \sum_{\lambda} \sum_{\lambda'} R_{12}(\lambda \rightarrow f) R_{22}(\lambda' \rightarrow f) \\
&\quad \times a(\lambda) a(\lambda') L_{\lambda} L_{\lambda'} T_{\lambda\lambda'} / E(\lambda) E(\lambda') \\
\overline{x'^2(f)} &\approx e^2 \sum_{\lambda} \sum_{\lambda'} R_{22}(\lambda \rightarrow f) R_{22}(\lambda' \rightarrow f) \\
&\quad \times a(\lambda) a(\lambda') L_{\lambda} L_{\lambda'} T_{\lambda\lambda'} / E(\lambda) E(\lambda')
\end{aligned} \tag{7}$$

where $T_{\lambda\lambda'}$ and $S_{a,m}$ are defined as

$$T_{\lambda\lambda'} \equiv \sum_m q_m S_{a,m}(\lambda) S_{a,m}(\lambda') / \sum_m q_m \tag{8}$$

$$S_{a,m}(\lambda) \equiv S_m(\lambda) - \sum_m q_m S_m(\lambda) / \sum_m q_m \tag{9}$$

Let's consider 'average' of emittance dilution over many linacs which have different random misalignment. Let ' $\langle \rangle$ ' denote average over different linacs.

$$\langle \Delta \varepsilon \rangle \approx \frac{1 + \alpha^2}{2\beta} \langle \overline{x^2} \rangle + \alpha \langle \overline{xx'} \rangle + \frac{\beta}{2} \langle \overline{x'^2} \rangle \tag{10}$$

In the case that a linac is divided into alignment units, each unit is aligned independently, $\langle a(\lambda) a(\lambda') \rangle$ vanishes if λ and λ' do not belong to the same unit.

$$\langle a(\lambda) a(\lambda') \rangle = \begin{cases} a_i^2 & (\lambda, \lambda' \in i) \\ 0 & (\lambda, \lambda' \notin i) \end{cases} \tag{11}$$

where r.m.s. of misalignment of i-th unit is a_i . From this,

$$\begin{aligned}
\langle \overline{x^2(f)} \rangle &\approx e^2 \sum_i a_i^2 \sum_{\lambda \in i} \sum_{\lambda' \in i} R_{12}(\lambda \rightarrow f) R_{12}(\lambda' \rightarrow f) L_{\lambda} L_{\lambda'} T_{\lambda\lambda'} / E(\lambda) E(\lambda') \\
\langle \overline{xx'(f)} \rangle &\approx e^2 \sum_i a_i^2 \sum_{\lambda \in i} \sum_{\lambda' \in i} R_{12}(\lambda \rightarrow f) R_{22}(\lambda' \rightarrow f) L_{\lambda} L_{\lambda'} T_{\lambda\lambda'} / E(\lambda) E(\lambda') \\
\langle \overline{x'^2(f)} \rangle &\approx e^2 \sum_i a_i^2 \sum_{\lambda \in i} \sum_{\lambda' \in i} R_{22}(\lambda \rightarrow f) R_{22}(\lambda' \rightarrow f) L_{\lambda} L_{\lambda'} T_{\lambda\lambda'} / E(\lambda) E(\lambda')
\end{aligned} \tag{12}$$

Emittance dilution is

$$\langle \Delta \varepsilon \rangle \approx e^2 \sum_i a_i^2 \sum_{\lambda \in i} \sum_{\lambda' \in i} B_{f,\lambda\lambda'} L_{\lambda} L_{\lambda'} T_{\lambda\lambda'} / E(\lambda) E(\lambda') \tag{13}$$

where

$$\begin{aligned}
B_{f,\lambda\lambda'} &\equiv \frac{1 + \alpha_f^2}{2\beta_f} R_{12}(\lambda \rightarrow f) R_{12}(\lambda' \rightarrow f) + \alpha_f R_{12}(\lambda \rightarrow f) R_{22}(\lambda' \rightarrow f) \\
&\quad + \frac{\beta_f}{2} R_{22}(\lambda \rightarrow f) R_{22}(\lambda' \rightarrow f)
\end{aligned} \tag{14}$$

Using equation (13), we can calculate average of emittance dilution without any random numbers. $B_{f,\lambda\lambda}$, L_λ and $E(\lambda)$ are obtained from the design parameters of the linac and $S_{a,m}(\lambda)$ from the wakefunction of accelerating structures. Though we need to use a computer for this calculation, required time is negligibly small compared with tracking simulations where many linacs with different seeds of random misalignments are simulated to reduce the statistical error.

Note that the essential assumption is that the beam oscillation is negligibly small compared with misalignment of structures.

2. Analytical expression

In the case that each alignment unit is enough small that transfer matrix can be treated as constant in a unit, a unit can be considered as a piece of structure or only one piece belongs to each alignment unit. Index λ can be replaced with i in equations in previous section.

Equation 11 becomes

$$\langle a(i)a(i') \rangle = \delta_{ii'} a_i^2 \quad (15)$$

and

$$\begin{aligned} \langle x^2(f) \rangle &\approx e^2 \sum_i a_i^2 R_{12}^2(i \rightarrow f) L_i^2 S_{rms}^2(i) / E_i^2 \\ \langle xx'(f) \rangle &\approx e^2 \sum_i a_i^2 R_{12}(i \rightarrow f) R_{22}(i \rightarrow f) L_i^2 S_{rms}^2(i) / E_i^2 \\ \langle x'^2(f) \rangle &\approx e^2 \sum_i a_i^2 R_{22}^2(i \rightarrow f) L_i^2 S_{rms}^2(i) / E_i^2 \end{aligned} \quad (16)$$

where

$$S_{rms}^2(i) = \sum_m q_m S_{a,m}^2(i) / \sum_m q_m \quad (17)$$

is r.m.s. of 'sum wake'.

Because

$$\frac{1 + \alpha_f^2}{2\beta_f} R_{12}^2(i \rightarrow f) + \alpha_f R_{12}(i \rightarrow f) R_{22}(i \rightarrow f) + \frac{\beta_f}{2} R_{22}^2(i \rightarrow f) = \frac{E_i \beta_i}{2E_f^2} \quad (18)$$

where E_i and β_i are beam energy and beta-function at i , emittance dilution is

$$\langle \Delta \epsilon \rangle \approx e^2 \sum_i \frac{\beta_i}{2E_f E_i} a_i^2 L_i^2 S_{rms}^2(i) \quad (19)$$

Assume all alignment unit have the same r.m.s. of misalignment, the same wakefunction and the same length as

$$a_i^2 = a^2 \quad (20)$$

$$S_{rms}^2(i) = S_{rms}^2 \quad (21)$$

$$L_i = L_a \quad (22)$$

Assume beta-function is smooth (continuous focusing) and depends on beam energy as

$$P_i = \beta_0 (E_i/E_0)^\alpha \quad (23)$$

where E_0 and β_0 are beam energy and beta-function at the beginning of the linac. Transforming the summation over units (index i) to continuous integration as

$$\sum_i L_i \rightarrow \int ds \rightarrow \int dE / g$$

where g is accelerating gradient, we get emittance dilution as

$$\langle \Delta \mathcal{E} \rangle \approx \frac{e^2 a^2 S_{rms}^2 \beta_0 L_a (E_f^\alpha - E_0^\alpha)}{2 \alpha E_f E_0^\alpha g} = \frac{e^2 a^2 S_{rms}^2 \beta_0 N_a L_a^2 (E_f^\alpha - E_0^\alpha)}{2 \alpha E_f E_0^\alpha (E_f - E_0)} \quad (24)$$

where N_a is the total number of units.

References

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