



Helical Undulator Radiation

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Abstract: Expressions for the energy spectrum, photon number spectrum, and polarization of radiation emitted from a helical undulator are presented. Photon number normalization is discussed. The effect of angular collimation of the radiation is shown. Matlab script files used in the calculations are listed. Algorithms of what is actually calculated are documented. The notation used closely follows that of NLC Note LCC-0085.

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rev, 3: 7/24/02

References:

- (1) J. C. Sheppard, *Planar Undulator Considerations*, NLC Note **LCC-0085**, July, 2002.
- (2) Brian M. Kincaid, *A short-period helical wiggler as an improved source of synchrotron radiation*, Journal of Applied Physics, **Vol. 48 No. 7**, July 1977, 2684-2691.
- (3) M. Born and E. Wolf, **Principles of Optics, 5th ed.**, Pergamon Press, New York, 1975, pp. 28-32 and 554-555.
- (4) H. Wiedemann, **Particle Accelerator Physics II**, Springer-Verlag, Berlin Heidelberg, 1995, Chapter 11.
- (5) K. Flöttmann, *Investigation Toward the Development of Polarized and Unpolarized High Intensity Positron Sources for Linear Colliders*, **DESY 93-161a**, November, 1993. [Note, most of what follows is presented and discussed in this reference].

Abstract

Expressions for the energy spectrum, photon number spectrum, and polarization of radiation emitted from a helical undulator are presented. Photon number normalization is discussed. The effect of angular collimation of the radiation is shown. Matlab script files used in the calculations are listed. Algorithms of what is actually calculated are documented. The notation used closely follows that of NLC Note **LCC-0085**.

Question: What's the energy spectrum, photon number spectrum, and polarization of radiation emitted from a helical undulator? What is the effect of angular collimation?

Basic Considerations

First, what is the amount of total radiated energy? Without explanation, at this time, the total radiated energy per meter of helical undulator per electron is given as ΔE ,

$$\Delta E = \frac{2}{3} r_c mc^2 \mathbf{g}^2 K^2 k_u^2 = 1450 \frac{E_e^2 (\text{GeV}) K^2}{I_u^2 (\text{cm})} \text{ eV/m/e}^- \quad (1)$$

where in E_e is the electron energy, I_u is the undulator period, and K is the undulator parameter which has the same definition as for the case of a planar undulator¹,

$$K = 9.344 B_0 (kG) I_u (m) \quad (2)$$

with B_0 being the magnetic field strength. Note that the total radiated energy per meter for a helical undulator is twice that of a planar undulator for the same values of E , K , and I_u , due to the average greater acceleration in the helical field.

The harmonic cutoff energies are given by E_{ch0}

$$E_{ch0} = h \frac{2\mathbf{g}^2 \hbar \mathbf{w}_u}{1 + K^2} = 9.497 \times 10^{-4} h \frac{E_e^2 (GeV)}{\mathbf{I}_u (cm) (1 + K^2)} MeV \quad (3)$$

where h is a natural number and $\mathbf{w}_u = 2\mathbf{p} c / \mathbf{I}_u$. Note that E_{ch0} is different for the helical case from the planar by the K^2 versus $K^2/2$ term in the denominator. This reduces E_{ch0} for given values of E_e, \mathbf{I}_u , and K in comparison (with a planar undulator).

Energy Spectrum, Photon Number Spectrum, and Polarization Spectrum

Equations 24 and 25 of reference (2) (see the Appendix) have been coded up in Matlab script files to calculate the angular spectrum ($dIdwT$), energy spectrum ($dIdw$ and $hsum$) and the third Stokes parameter³, s_3 (pol), as a functions of emission angle, \mathbf{q} and of normalized frequency ($ww10$). The notation developed in reference (1) is typically used below. [Note, the methodology in reference (4) has been followed to develop the expressions for determining s_3 .] As with the case of the planar undulator¹ the shape of the energy, photon number, and also polarization as functions of the normalized frequency depend only on the undulator parameter, K .

The energy spectrum, $ampl(h, ww10)$, is calculated as a function of $ww10 = \mathbf{w} / \mathbf{w}_{10}$, $\mathbf{w}_{10} = E_{c10} / \hbar$ and $\mathbf{w} = h 2\mathbf{g}^2 \mathbf{w}_u / (1 + K^2 + \mathbf{g}^2 \mathbf{q}^2)$, h is the harmonic number and \mathbf{q} is the "viewing" angle. Each harmonic is calculated separately. The photon helicity, $pol(h, ww10)$ is determined for each \mathbf{w} by computing s_3 . These are also calculated for each harmonic separately. The full energy spectrum $hsum$ is simply the sum of the individual harmonics, $hsum = \sum_h ampl(h, ww10)$. The photon number spectrum is given by $nsum = hsum / ww10$. Finally, the composite polarization is given as $psum$,

$$psum = \left(\sum_h pol(h, ww10) \cdot ampl(h, ww10) / ww10 \right) / nsum \quad (4)$$

Figures 1, 2, and 3 show the energy spectrum, photon number spectrum, and circular polarization for a $K=1$ helical undulator. For the case of $\mathbf{I}_u = 1 cm$ and beam energy of $E_0 = 150 GeV$, $E_{c10} = 10.7 MeV$. Following the notation developed in reference (1), the average photon energy, ${}_m E_{avg}$ is given as

$${}_m E_{avg} = E_{c10} \times \frac{\sum hsum}{\sum hsum / ww10} \quad (5)$$

For $K = 1$ helical undulator radiation, $I_m E_{avg} = 0.84 \times E_{c10}$. The number of photons radiated per meter, ΔN_{in} , is found by taking the ratio $\Delta E / I_m E_{avg}$ (expressions (1), (3) and (5)):

$$\Delta N_{in} = \frac{\Delta E}{I_m E_{avg}} = 1.53 \frac{K^2 (1 + K^2)}{I_u (cm)} \times \frac{\sum hsum./ ww10}{\sum hsum} \text{ photons / m / } e^- . \quad (6)$$

The factor $H(K) = \frac{\sum hsum./ ww10}{\sum hsum}$ which enters into (5) and (6) is a smoothly varying function of K with values from $2 \rightarrow 0.5$ over the range of $0.001 \leq K \leq 1.5$. For larger K , more harmonics need to be included in the calculations (40 harmonics have been used in this exercise). Figure 4 displays the $H(K)$ factors: $H(K)$, $1/H(K)$, and $K^2 (1 + K^2) H(K)$ versus K . For illustrative purposes, $\Delta N_{in}(K)$ has been fit to a second order polynomial in K over the range of $0 \leq K \leq 2$; for detailed evaluation, expression (6) should be used.

$$\Delta N_{in}(K) \approx (1.80 \times K^2 + 0.80 \times K - 0.07) / I_u (cm) \text{ photons / m / } e^- . \quad (7)$$

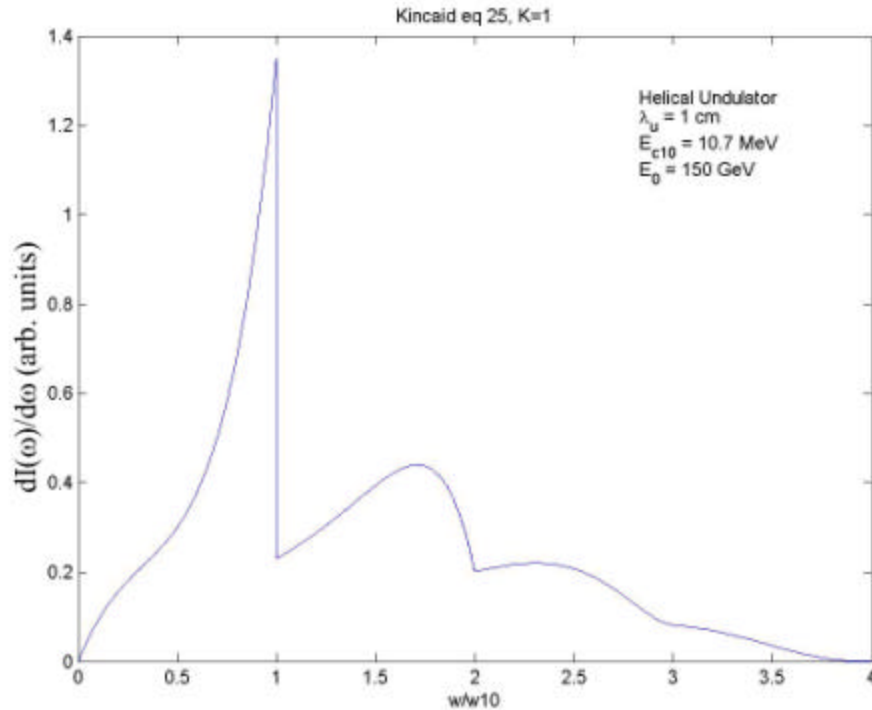


Figure 1: Helical undulator radiation energy spectrum for $K = 1$.

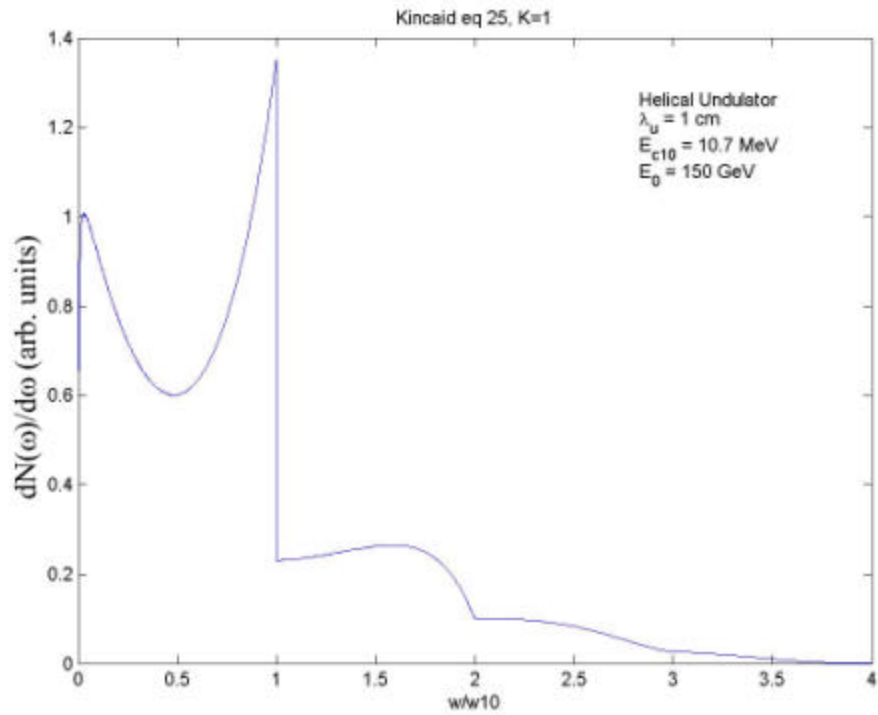


Figure 2: Helical undulator radiation photon number spectrum for $K = 1$.

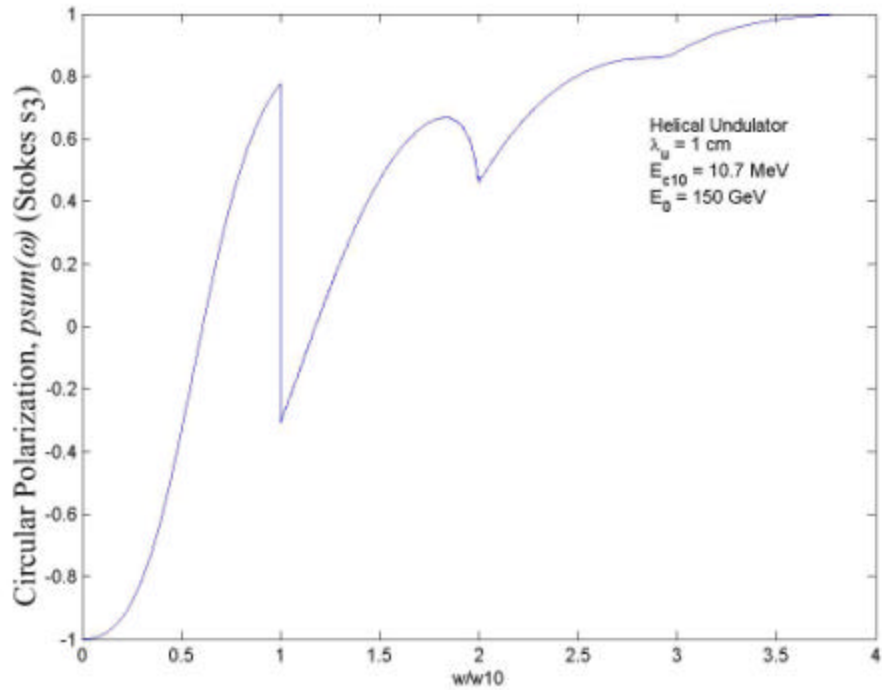


Figure 3: Helical undulator radiation circular polarization for $K = 1$. Note, only the first 4 harmonics have been included in the summation.

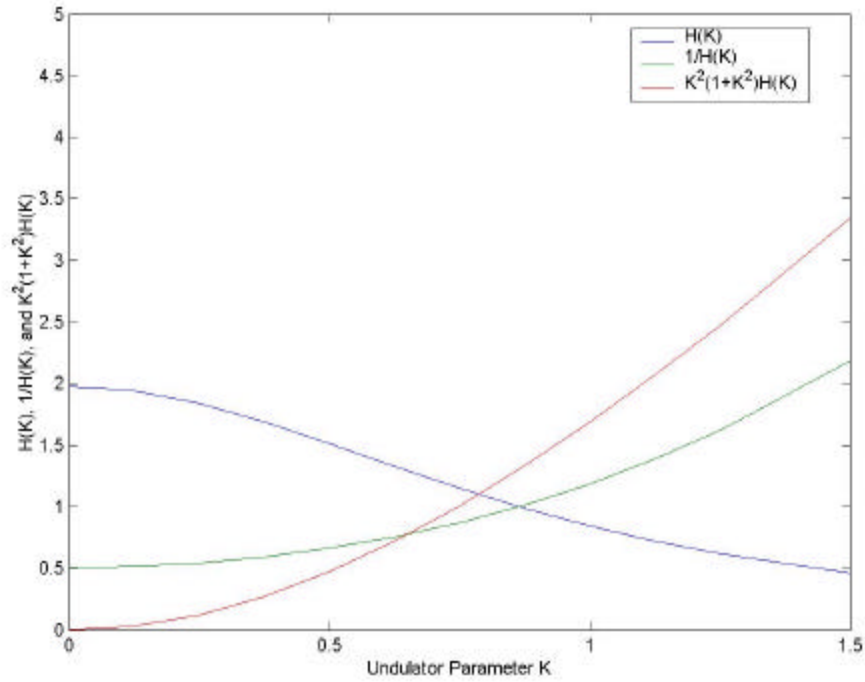


Figure 4.: $H(K) = \frac{\sum hsum./ww10}{\sum hsum}$ factors versus K . The first 40 harmonics have been used in the evaluation.

To evaluate the number of photons emitted per meter from equation (6), a relatively large number of harmonics must be included in the calculation. As seen in Figure 5, the larger the value of K , the larger the number of harmonics that must be included. For all values of K , the larger the number of harmonics the better the result. Figure 5 shows expression (6) for a range of K and for different number of included harmonics, h . The "+" points in the figure have been taken from reference (5) wherein evidently only the first 7 harmonics have been included. Figure 6 is a replot of Figure 5 over a reduced range of K and only displays the case for 40 harmonics and the quadratic fit, equation (7). From Figure 7, for $K = 1$ and $I_u = 1 \text{ cm}$, $\Delta N_m = 2.6 \text{ photons/m/e}^-$. As noted above, twice as much energy is radiated per meter in a helical versus a planar undulator for the same values of K and I_u . Since the photon energies (helical/planar) scale as $(1 + K^2/2)/(1 + K^2)$, there will be more than twice as many photons produced per meter for a helical undulator. When applied to the purpose of making positrons, it is noted that positron production in general scales linearly with total incident energy, the product of photon number times photon energy. Therefore, about twice as many positrons are generated for a fixed length of undulator when a helical rather than a planar device is used.

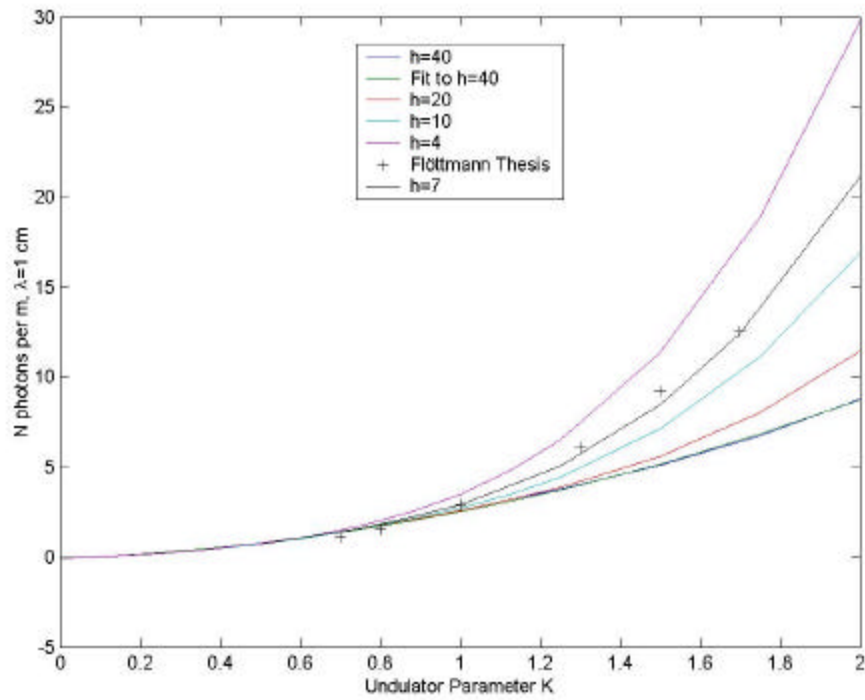


Figure 5.: Number of photons radiated per meter of undulator (equation (6)) as a function of K . The effect on the calculation of including different numbers of harmonics, h , is seen; the more harmonics used, the better is the result.

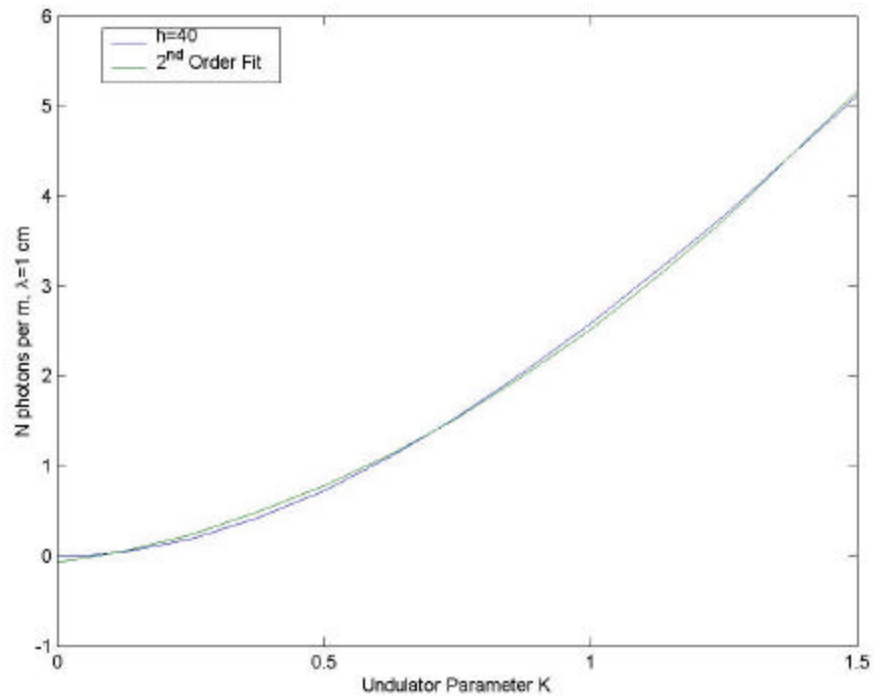


Figure 6.: Number of photons radiated per meter of undulator (equation (6)) as a function of K . 40 harmonics have been included in the fit; also shown is the quadratic fit, equation (7).

Emission Angle Considerations

In the case of a large number of undulator periods, the photons in a particular harmonic are segregated in frequency, $w(h, \mathbf{q})$, by emission angle, \mathbf{q} :

$$w(h, \mathbf{q}) = h \frac{2g^2 w_u}{(1 + K^2 + g^2 q^2)}. \quad (8)$$

Figures 7 and 8 show the frequency and energy of the emitted radiation as a function of emission angle. The strong correlations amongst angle, frequency, and polarization will allow one to increase the polarization of collected radiation through angular collimation.

Figure 9 shows the emitted flux as a function of angle for the sum over the harmonics and the integral with respect to the emission angle of the sum over the harmonics. As seen in Figure 9, an angular cut that excludes angles $g\mathbf{q} \geq 1.414$ reduces the total flux to 87% of the total but results in greater overall polarization of the remaining photons, as shown in Figure 10. Figures 11a and 11b show the effect of collimation on the frequency spectrum.

For the curious minded, the transformation from Kincaid's expression (24), $\frac{dI}{d\mathbf{q}}$, of the radiation pattern with respect to angle to expression (25), $\frac{dI}{dw}$, which is the frequency spectrum of the radiation is made deceptively simple under the assumption of a large number of undulator periods. With this assumption the segregation of w wrt \mathbf{q} is complete and the transformation is

$$\frac{dI}{d\mathbf{q}}(\mathbf{q}) = \frac{1}{\sin(\mathbf{q})} \left| \frac{\partial w}{\partial \mathbf{q}} \right| \frac{dI}{dw}(w(h, \mathbf{q})) \cong \frac{1}{\mathbf{q}} \left| \frac{\partial w}{\partial \mathbf{q}} \right| \frac{dI}{dw} = \frac{w^2(h, \mathbf{q})}{hw_u} \frac{dI}{dw}(w(h, \mathbf{q})) \quad (9)$$

where $w(h, \mathbf{q})$ is given by equation (8), $w_u = 2p c/l_u$, and $\frac{dI}{dw}(w)$ is given by Kincaid's expression (25)². In (9), the sum over the various harmonics is implicit while the harmonic number h is explicitly shown.

Pertinent Matlab Script Files

Equations 24 and 25 of reference (2) have been coded up in Z:\Positrons\Polarized Positrons\helicalTheta.m and \helicalspect.m. \helicalTheta.m calculates the angular spectrum ($dIdwT$) and the third Stokes parameter s_3 , as a functions of emission angle, ($ThetaT$). \helicalspect.m calculates the energy spectrum ($dIdw$ and $hsum$) and s_3 (pol) as a functions of normalized frequency ($wwl0$). \helicalspect.m also calculates the angular spectrum ($dIdwdt$) from $dIdw$ as per expression (8) herein. [Note, the methodology in reference (4) has been followed to develop the expressions in \helicalTheta.m and \helicalspect.m for determining s_3 .] See also Z:\Positrons\Polarized Positrons\eq1149df.m wherein the nomenclature is first developed. The routines Z:\Positrons\Polarized Positrons\h_of_k.m and \h_of_k_plots.m are used in conjunction with helicalspect.m to determine $H(K)$ and the fit of expression (7). In \h_of_k.m,

$\frac{\sum h_{sum} / w w_{10}}{\sum h_{sum}}$ is replaced by $\frac{\sum_h \sum_w dI d w(h, \mathbf{w}) / (\mathbf{w} / w_{10})}{\sum_h \sum_w h \times dI d w(h, \mathbf{w})}$ so as to allow for

examination of the effect of changing the number of harmonics included in the summation. Each pass through `.h_of_k.m` requires a run of `.helicalspect` for a given value of K .

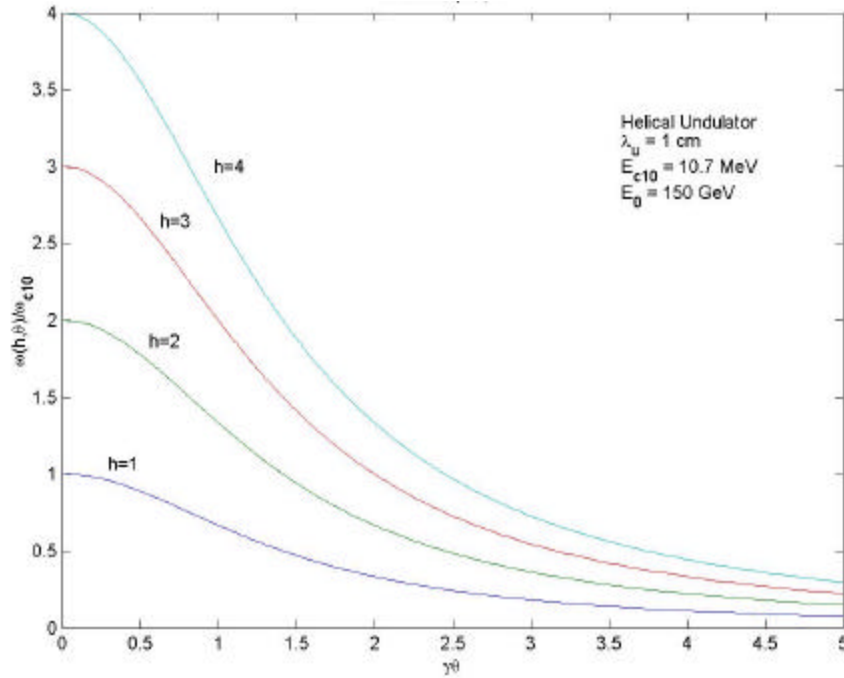


Figure 7.: Frequency of emitted radiation from a long helical undulator as a function of emission angle. The first 4 harmonics are shown.

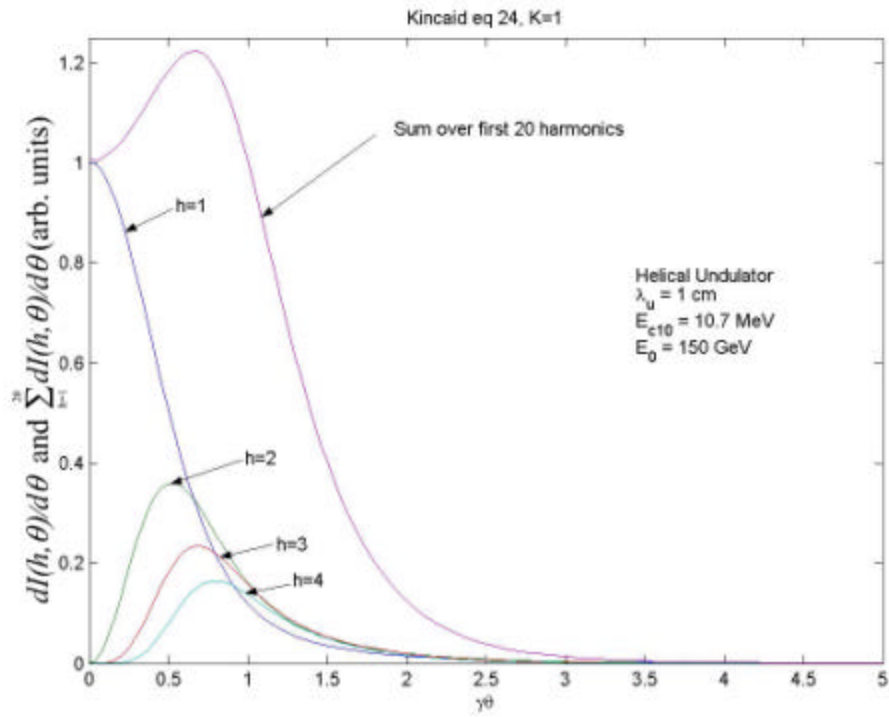


Figure 8.: Helical undulator emitted flux as a function of emission angle.

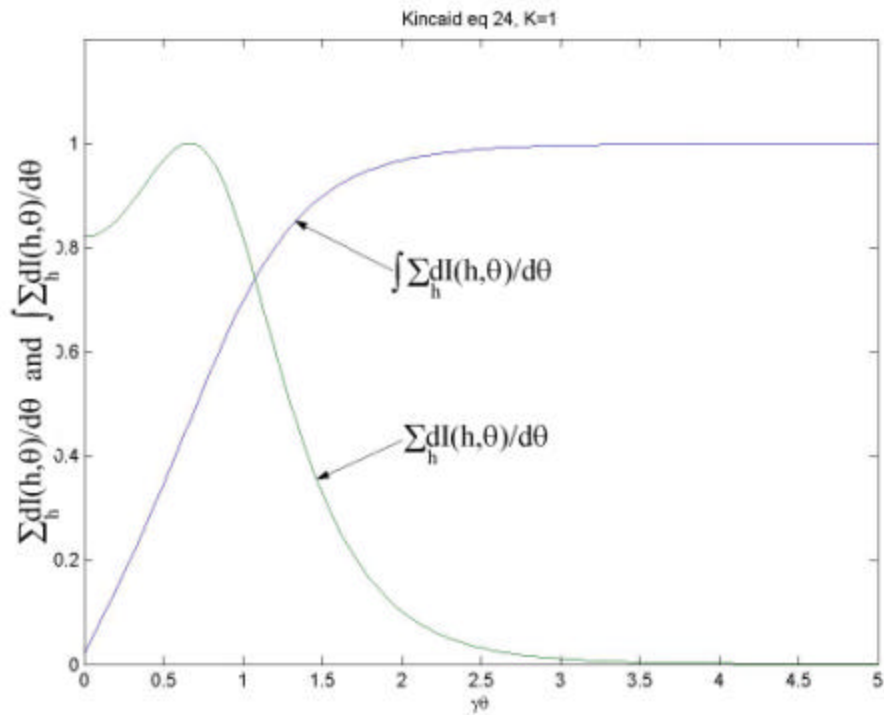


Figure 9.: The emitted flux summed over 20 harmonics versus the emission angle and the integral of the harmonic sum with respect to the angle.

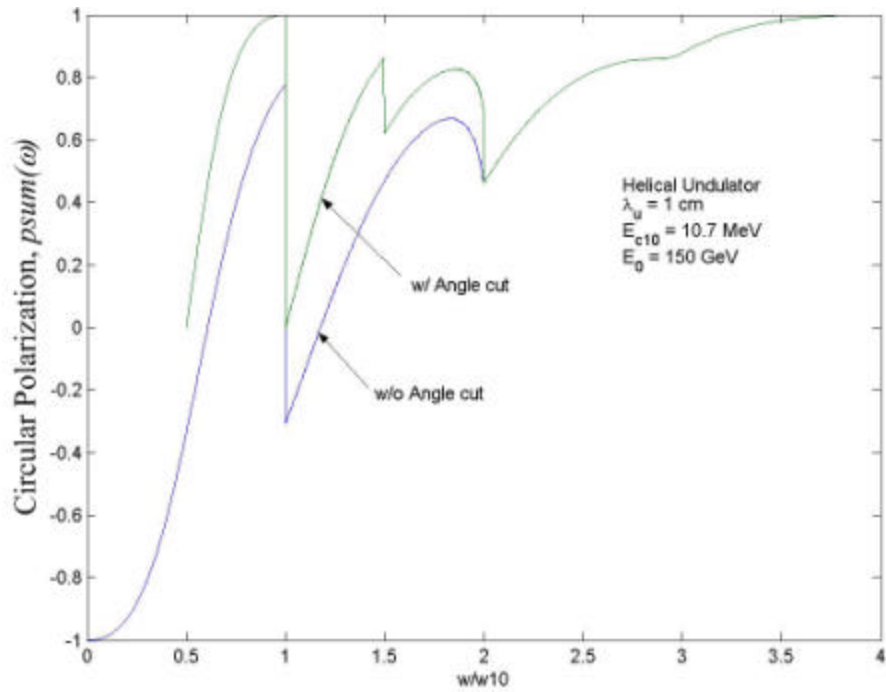


Figure 10.: Circular polarization of the emitted radiation with (green) and without (blue) an angle cut at $gq \geq 1.414$. The termination of the green curve at $w/w_{10} = 0.5$ reflects the lack of flux due to the cut (see figure 8).

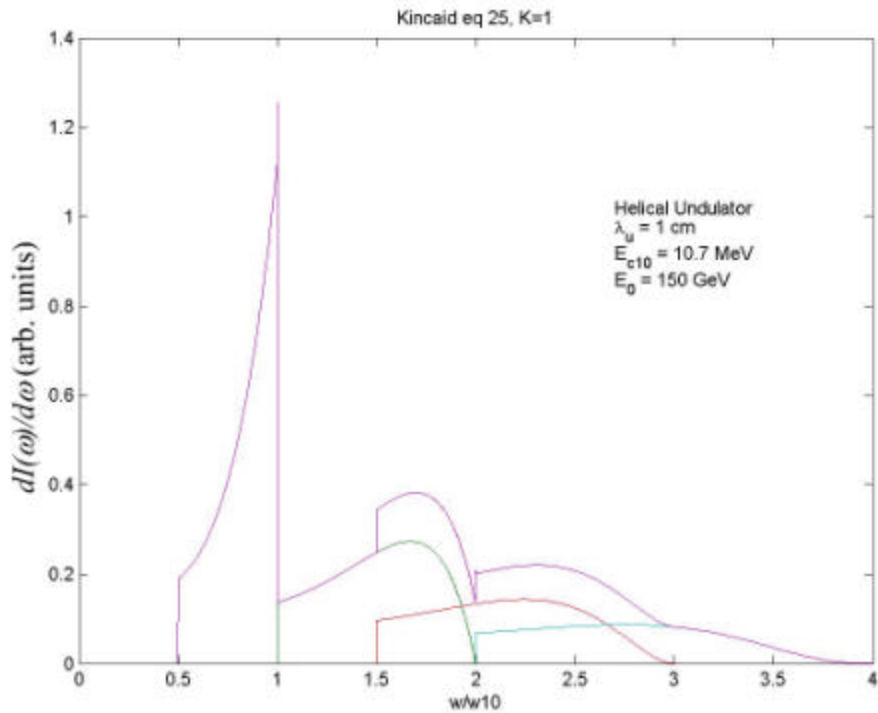


Figure 11a.: The flux emitted from a helical undulator, $K = 1$, after an angle cut that excludes $gq \geq 1.414$. The sum over the 4 harmonics is shown (purple) along with the individual harmonics.

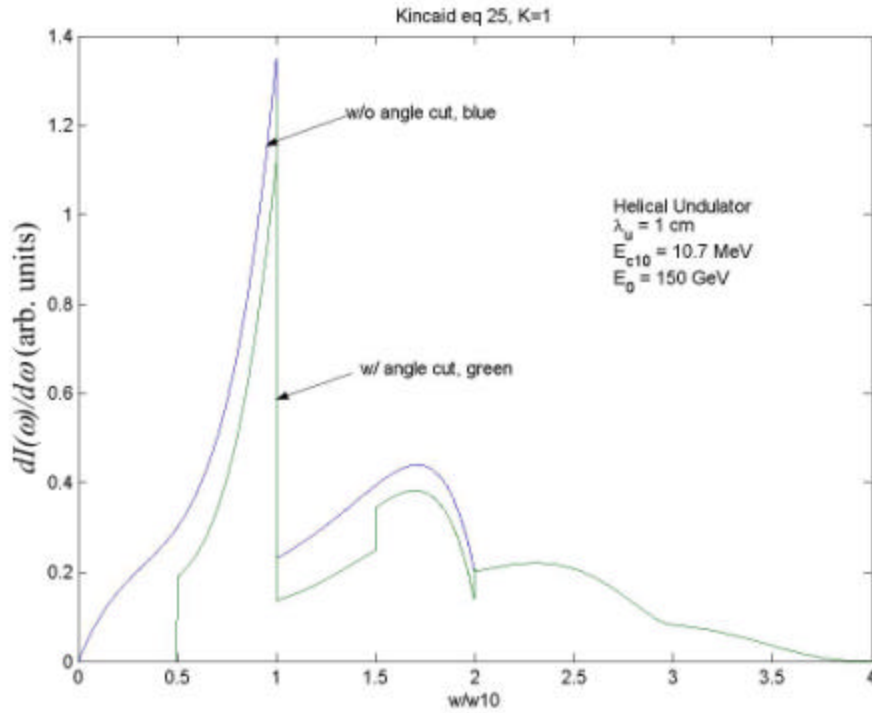


Figure 11b. The flux emitted from a helical undulator, $K = 1$, after an angle cut that excludes $gq \geq 1.414$ (green). The flux without the angle cut (blue) is also shown for comparison.

Appendix: Kincaid's equations (24) and (25)

Kincaid's equations (24) and (25) of reference (2) are shown below. The notation has been changed slightly, most notably $\frac{dI}{d\mathbf{w}} \equiv I(\mathbf{w})$, where $I(\mathbf{w})$ is Kincaid's notation for the energy spectrum. In (A1) and (A2), N is the number of undulator periods. (A1) and (A2) have been converted to MKS through division by $4pe_0$.

Equation (24), reference (2):

$$\frac{dI}{d\Omega} \equiv \frac{dW}{d\Omega} = \frac{2Nq^2 \mathbf{w}_u \mathbf{g}^4}{pe_0 c (1 + K^2 + \mathbf{g}^2 \mathbf{q}^2)^3} \sum_{h=1}^{\infty} h^2 \left[J_h'^2(x_h) + \left(\frac{gq}{K} - \frac{h}{x_h} \right)^2 J_h^2(x_h) \right] J_{-sr}, \quad (\text{A1})$$

where $x_h = 2Khgq / (1 + K^2 + \mathbf{g}^2 \mathbf{q}^2)$.

Equation (25), reference (2) (note: the \mathbf{g} in eq. (25) should be an r , see ref. (2) appendix):

$$\frac{dI}{d\mathbf{w}} \equiv I(\mathbf{w}) = \frac{Nq^2 K^2 r}{e_0 c} \sum_{h=1}^{\infty} \left[J_h'^2(x_h) + \left(\frac{\mathbf{a}_h}{K} - \frac{h}{x_h} \right)^2 J_h^2(x_h) \right] u(\mathbf{a}_h^2) J_{-s}, \quad (\text{A2})$$

where $\mathbf{a}_h^2 = h/r - 1 - K^2$, $x_h = 2Kra_h$, $r = \mathbf{w}/2\mathbf{g}^2\mathbf{w}_u$, and $u(\mathbf{a}_h^2)$ is a unit step function.

Note in (A1) and (A2): $J'_h(x) \equiv \frac{d}{dx} J_h(x) = \frac{1}{2}(J_{h-1}(x) - J_{h+1}(x))$.