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Energy Loss and Energy Spread Growth In a Planar Undulator

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Energy Loss and Energy Spread Growth
in a Planar Undulator

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References:

1. J. C. Sheppard, *Planar Undulator Considerations*, NLC Note LCC-0085.
2. E. L. Saldin, et al., *Calculation of energy diffusion in an electron beam due to quantum fluctuations of undulator radiation*, **Nucl. Instr. & Meth. A381** (1996) 545-547.
3. P. Beckmann, **Elements of Applied Probability Theory**, Harcourt, Brace & World, Inc. (1968), 100ff.
4. **TESLA TDR II Accelerators**, chapters 4 and 5.

Abstract

The change in beam energy spread due to transmission through a long, planar undulator is calculated. This change is shown to be gaussian as expected from the central limit theorem and large number of photons emitted per electron. These results are compared with Saldin *et al.* [2] expressions. Numerical results for the case of the TESLA beam and for an NLC beam are given.

Question: What is the change in energy spread due to a planar undulator?

For an energy loss d , the electron beam energy E is given as

$$E = E_0 - d \tag{1}$$

wherein E_0 is the initial beam energy. The average beam energy is simply

$$\langle E \rangle = \langle E_0 - d \rangle = \langle E_0 \rangle - \langle d \rangle \tag{2}$$

where $< d >$ is the mean radiated energy. The rms energy spread s_E is

$$s_E^2 = \left\langle \left(E_0 - d \right)^2 - \left\langle E_0 - d \right\rangle^2 \right\rangle$$

$$= \left(\left\langle E_0^2 \right\rangle - \left\langle E_0 \right\rangle^2 \right) + \left(\left\langle d^2 \right\rangle - \left\langle d \right\rangle^2 \right) - 2\left(\left\langle E_0 d \right\rangle - \left\langle E_0 \right\rangle \left\langle d \right\rangle \right). \tag{3}$$

For small initial energy spread in E_0, the last term in the rhs of (3) can be set to zero (see Appendix A). Thus the energy spread can be written as

$$s_E^2 \cong \left(\left\langle E_0^2\right\rangle - \left\langle E_0\right\rangle^2\right) + \left(\left\langle d^2\right\rangle - \left\langle d\right\rangle^2\right) = s_{E_0}^2 + s_d^2 \qquad (4)$$

where s_{E_0} is the initial rms energy spread of the incident beam and s_d is the additional energy spread due to the undulator.

Now figure out $s_d^2 = \left(\left\langle d^2\right\rangle - \left\langle d\right\rangle^2\right)$. The photon number spectrum $n(w)$ is shown in figure 1 for a $K = 1$ undulator in the case of a large number of undulator periods, $g \square 1$, and small initial emittances ($s_{x'}, s_{y'} \square \frac{1}{g}$ and $\frac{s_{E_0}}{E_0} \square 1$). With proper normalization

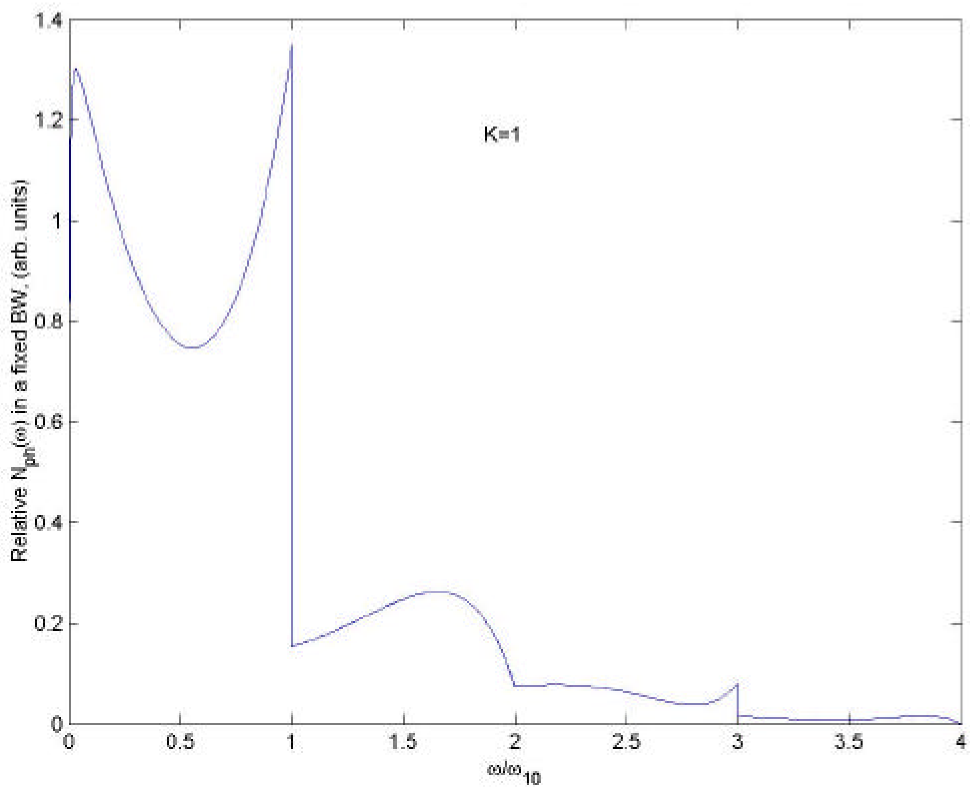

Figure 1: The universal photon number spectrum, $n(w/w_{10}) = hsum(w/w_{10})./(w/w_{10})$

$\left(\int_0^{\infty} dw\, n(w) = 1\right)$ $n(w)$ is the probability that an emitted photon has a frequency in the interval $w \rightarrow w + dw$. The mean energy of the emitted photons is given as

$$\hbar\left\langle w\right\rangle = \hbar\int_0^{\infty} dw\, w\, n(w) \qquad (5)$$

and the mean square energy is

$$\hbar^2 \left\langle w^2 \right\rangle = \hbar^2 \int_0^\infty dw w^2 n(w) \qquad (6)$$

For an undulator of length L_u, the average number of emitted photons per electron from [1] is N_{ph},

$$N_{ph} = {}_eY_{hu} L_u . \qquad (7)$$

Expressions (5), (6), and (7) are combined to yield [3]

$$\left\langle d \right\rangle = N_{ph} \hbar \left\langle w \right\rangle , \qquad (8a)$$

$$\left(\left\langle d^2 \right\rangle - \left\langle d \right\rangle^2 \right) = N_{ph} \hbar^2 \left(\left\langle w^2 \right\rangle - \left\langle w \right\rangle^2 \right), \qquad (8b)$$

$$\left\langle d'^2 \right\rangle = N_{ph} \hbar^2 \left\langle w^2 \right\rangle . \qquad (8c)$$

So what are the values (reference Z:positorns/polarized positrons/tesla150_250wrkspc.m Matlab workspace).

For $K = 1$, the photon energy spectrum $hsum(ww10)$ is shown in figure 2 (note: this is what is calculated using Matlab file Z:/positrons/polarized positrons/tesla150_250wrkspc and ./1149df.m).

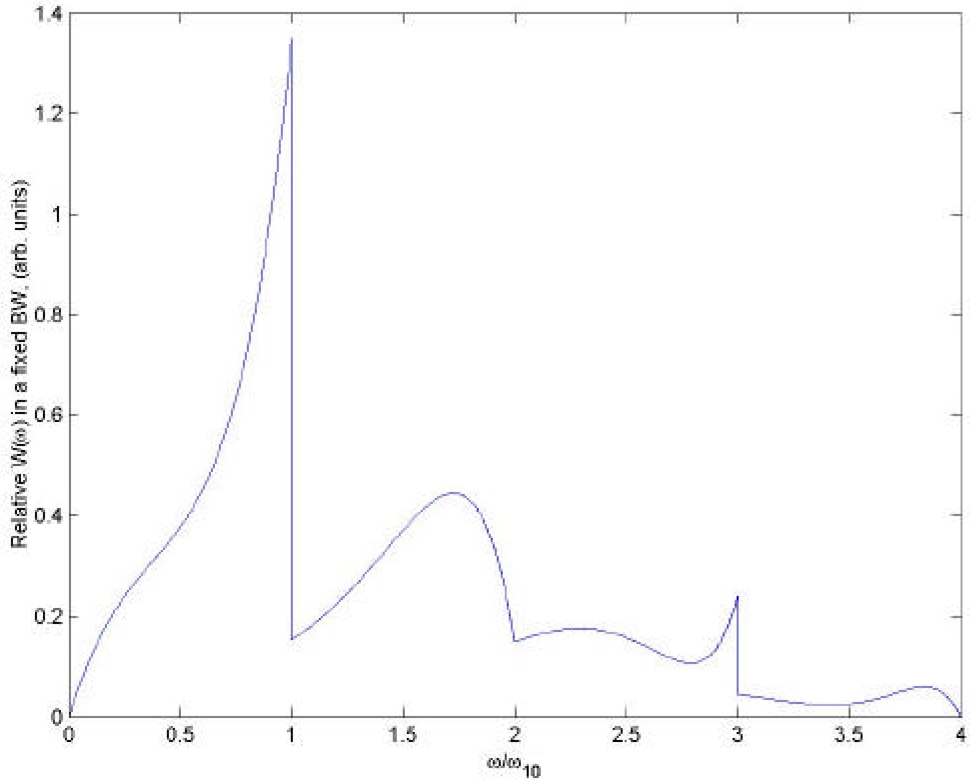

Figure 2: The universal undulator energy spectrum, $hsum(w/w_{10})$.

Note that

$$ww10 = \frac{w}{w_{10}} \tag{9}$$

where $w_{10} = E_{c10}/\hbar$, the cutoff frequency (rad/s) if the first harmonic.

Using $hsum(ww10)$, the integrals (5) and (6) are replaced with sums:

$$\hbar\int_0^\infty dww\,n(w) = E_{c10} \times \frac{\sum hsum}{\sum hsum./ww10} = E_{c10} \times 0.791 \tag{10a}$$

and

$$\hbar^2\int_0^\infty dww^2 n(w) = E_{c10}^2 \times \frac{\sum hsum.*ww10}{\sum hsum./ww10} = E_{c10}^2 \times 1.075 \tag{10b}$$

Finally,

$$\langle d \rangle = 0.791 N_{ph} E_{c10} , \tag{11a}$$

$$s_d^2 = \left(\langle d^2 \rangle - \langle d \rangle^2 \right) = N_{ph} E_{c10}^2 \left(1.075 - 0.791^2 \right) = 0.45 N_{ph} E_{c10}^2 . \tag{11b}$$

Also note,

$$\langle d'^2 \rangle = 1.075 N_{ph} E_{c10}^2 . \tag{11c}$$

From reference [1], $E_{c10} = 28 MeV$ (see equation (7) ref. [1]), and $N_{ph} = 135$ (see equation (12) ref. [1]). The values are:

$$\langle d \rangle = 2.99 GeV , \tag{12a}$$

$$s_d^2 = 4.8 \times 10^4 MeV^2 . \tag{12b}$$

$$\langle d'^2 \rangle = 1.1 \times 10^5 MeV^2 . \tag{12c}$$

For $E_0 = 250 GeV$, $\langle d \rangle / E_0 = 1.2\%$, $s_d / E_0 = 0.09\%$, and $\langle d'^2 \rangle^{1/2} / E_0 = 0.13\%$.

How does this compare to Saldin [2]? For an undulator of length L_u, equation (7) of ref. [2] becomes

$$\langle (dg)^2 \rangle = \frac{7}{15} \lambdabar_c r_e g^4 k_w^3 K^2 F(K) L_u \tag{13}$$

with $\lambdabar_c = \hbar / mc = 3.86 \times 10^{-13} m$, $r_e = 2.818 \times 10^{-15} m$, $k_w = 2p / l_w$, $K =$ the undulator parameter (equation (3) ref. [1]), and $F(K)$ is give is [2] as

$$F(K) = 1.20 K + \frac{1}{1 + 1.33K + 0.40K^2} . \tag{14}$$

For our case recall $E_0 = 250 GeV$, $l_w = 1.42 cm$, $K = 1$, and $L_u = 135 m$, equation (14) gives

$$\frac{\langle (dg)^2 \rangle^{1/2}}{(E_0 / m_o c^2)} = 0.15\% . \tag{15}$$

This result is curiously similar to (12c) (the ratio of (15)/(12c)=1.106). As well is should since we are both starting from the same point. The details of the undulator spectrum are buried in Saldin's *F(K)*. I haven't decided whether or not it is worthwhile trying to root out the discrepancy. The TRC references Saldin and claims that the energy spread of the electron beam increases from 0.05% to 0.15% due to the undulator. It would appear that the TESLA TDR [4] takes expression (13) rather than calculating the rms. This seems to be an oversight.

My conclusion is that the foregoing is the proper way to figure out the energy spread due to a long undulator. Equating the result of equation (7) in the Saldin reference [2] to the rms energy spread overestimates the effect by ignoring the mean.

Radiated Energy Spectrum

The total radiated energy per electron is due to the sum of the energies of the individual photons. On average there are $N_{ph} = E_{rad} / \hbar \langle w \rangle$ which equals 135 in this exercise. The distribution, $N_e(E)$ of the total radiated energies for an ensemble of electrons is expected to be gaussian

$$N_e(E) = \frac{1}{s_w \left(2p N_{ph}\right)} \exp\left(\frac{-\left(E - N_{ph}\hbar\langle w \rangle\right)^2}{2N_{ph} s_w^2} \right) \tag{16}$$

where

$$s_w^2 = \hbar^2 \langle w^2 \rangle - \hbar^2 \langle w \rangle^2 \tag{17}$$

Figure 3 is an overlay of equation (16) and a simulation in which energies of N_{ph} photons randomly drawn from the undulator photon number distribution (see figure 1) are summed for 1000 cases (see Z:/positrons/polarized positrons/photonspec.m Matlab file). The summations are shown as a staircase plot. In figure 3,

$$s_w^2 = 0.45 \times E_{c10}^2 \tag{18}$$

(compare with equation (11b)), $E_{c10} = 28MeV$ and $N_{ph} = 135$. $N_{ph}\hbar\langle w \rangle$ has been set equal to the mean of the staircase distribution. There is a discrepancy of about 2% in the means. This is due to how the binning is done. Also the amplitude of the gaussian has been normalized to give the proper total number of electrons when integrated. The agreement of simulation with the expected gaussian looks quite good, as expected.

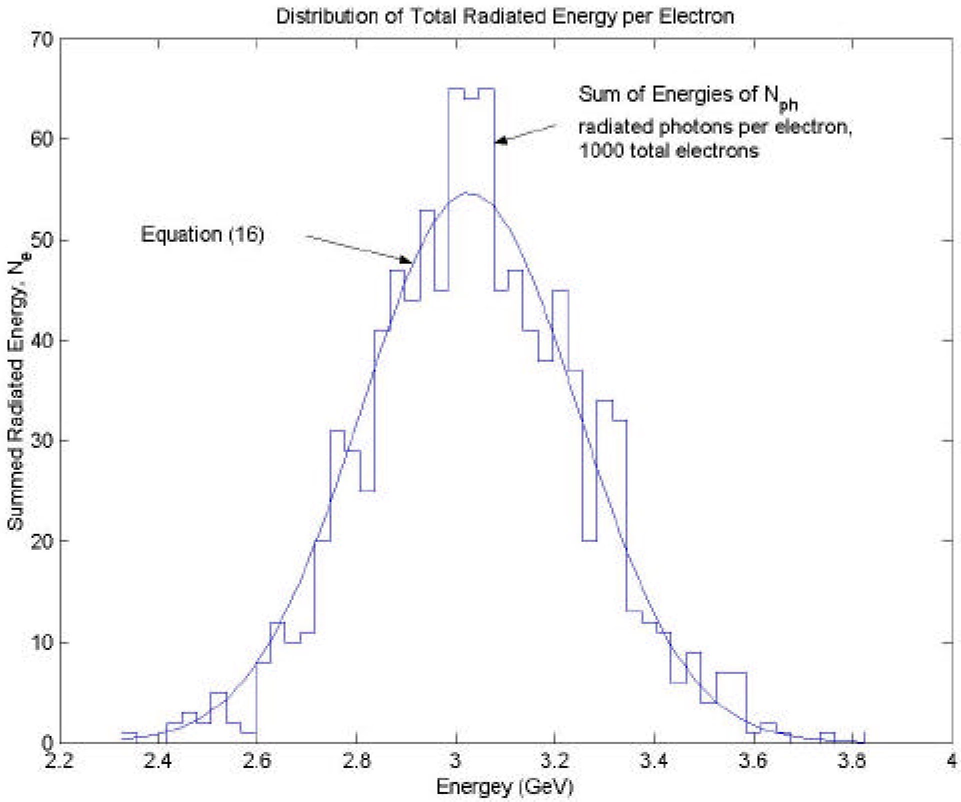

Figure 3: The distribution of total radiated energy from individual electrons.

Concluding Comments

A few words on scaling. For a given undulator, the mean radiated energy varies as $\gamma^2 L_u$ and the width of the distribution varies as $\gamma^2 L_u^{1/2}$. As expected, the distribution is gaussian. As seen in the Saldin formulation, the mean and variance also depend on the undulator period, λ_w and the undulator parameter K. The approach outlined here is in complete agreement with the Saldin paper. It is a straightforward exercise to rewrite the expressions in terms of $\gamma, k_{w,} K, L_u$ to get the same proportionality as [2].

For the NLC, the growth in energy spread is small in comparison to the expected $\sigma_E \approx 0.5\%$ at the 150 GeV point. For TESLA, the energy spread due to the undulator will increase by a factor of 2, not the factor of 3 stated in the TDR. The next task is to figure out the emittance growth induced by the dispersion in the undulator.

Appendix A

How Large is $-2\left(\langle E_0\delta\rangle - \langle E_0\rangle\langle\delta\rangle\right)$?

Starting with the first term, recall that $d = N_{ph}\hbar w$ and $w \propto E_0^2$ so that the first term can be written as

$$\langle E_0 d \rangle = \frac{\langle E_0^3 d' \rangle}{\langle E_0 \rangle^2} \tag{A1}$$

where $d' = d\,\langle E_0 \rangle^2 / E_0^2$. Since the E_0 dependence in d has been removed in d', **A1** can be rewritten as

$$\langle E_0 d \rangle = \frac{\langle E_0^3 \rangle \langle d' \rangle}{\langle E_0 \rangle^2} \equiv \frac{\langle E_0^3 \rangle}{\langle E_0 \rangle^2} \langle d \rangle . \tag{A2}$$

E_0 has a normal distribution, $p_{E_0}(E)$,

$$p_{E_0}(E) = \frac{1}{s_{E_0}\sqrt{2p}} \exp\left(\frac{-\left(E - \langle E_0 \rangle\right)^2}{2s_{E_0}} \right). \tag{A3}$$

$\langle E_0^3 \rangle$ is evaluated as follows,

$$\langle E_0^3 \rangle = \int_{-\infty}^{+\infty} dE E^3 p_{E_0}(E) \tag{A4a}$$

$$= \int_{-\infty}^{+\infty} dE \left(E + \langle E_0 \rangle\right)^3 p_{E_0}\left(E + \langle E_0 \rangle\right) \tag{A4b}$$

$$= \int_{-\infty}^{+\infty} dE \left(E^3 + 3\langle E_0 \rangle E^2 + 3\langle E_0 \rangle^2 E + \langle E_0 \rangle^3\right) p_{E_0}\left(E + \langle E_0 \rangle\right). \tag{A4c}$$

A4c is evaluated by inspection noting that the first and third terms are odd and therefore integrate to zero. Thus

$$\langle E_0^3 \rangle = \langle E_0 \rangle^3 + 3\langle E_0 \rangle s_{E_0}^2 . \tag{A4d}$$

A2 becomes

$$\left\langle E_0 d \right\rangle = \frac{\left\langle E_0 \right\rangle^3 + 3\left\langle E_0 \right\rangle s_{E_0}^2}{\left\langle E_0 \right\rangle^2} \left\langle d \right\rangle = \left(1 + 3\frac{s_{E_0}^2}{\left\langle E_0 \right\rangle^2} \right) \left\langle E_0 \right\rangle \left\langle d \right\rangle . \qquad \textbf{(A5)}$$

The dropped term in equation (3) is then

$$-2\left(\left\langle E_0 d \right\rangle - \left\langle E_0 \right\rangle \left\langle d \right\rangle \right) = -6\frac{s_{E_0}^2}{\left\langle E_0 \right\rangle^2} \left\langle E_0 \right\rangle \left\langle d \right\rangle . \qquad \textbf{(A6)}$$

This term needs to be evaluated in the context of the full beam energy spread as given by equation (3). The numbers for the NLC are $\left\langle E_0 \right\rangle = 250 GeV$, $\left\langle d \right\rangle = 3GeV$, $s_{E_0} / \left\langle E_0 \right\rangle \approx 0.5\%$, and from above $s_d / E_0 = 0.09\%$. Equation (3) gives

$$\frac{s_E}{\left\langle E_0 \right\rangle} = 0.49\%$$

while equation (4) gives

$$\frac{s_E}{\left\langle E_0 \right\rangle} = 0.51\% .$$

For the TESLA case, $\left\langle E_0 \right\rangle = 250 GeV$, $\left\langle d \right\rangle = 3GeV$, $s_{E_0} / \left\langle E_0 \right\rangle \approx 0.05\%$, and from above $s_d / E_0 = 0.09\%$. Equation (3) gives

$$\frac{s_E}{\left\langle E_0 \right\rangle} = 0.10\%$$

while equation (4) gives

$$\frac{s_E}{\left\langle E_0 \right\rangle} = 0.10\% .$$

Note that the dropped term does not change the answers by much. It seems a little curious that the dropped term decreases the overall energy spread in the NLC case. This could be straightforward radiation damping in that the higher energy electrons radiate away more than the lower energy electrons. Or maybe there is something to be considered in the energy dependence of d which has been ignored but may compensate. In any event, the effect is small and will be ignored for the time being. It gets picked naturally when the effect of the initial beam energy spread $s_{E_0} / \left\langle E_0 \right\rangle$ is included in the undulator spectrum.