OVERVIEW

1) Potential Functions, Complex Analysis, Analytic Functions

2) Direct/Indirect Flux Method, Excess Flux, Schwarz-Christoffel Transforms

3) Generic Hybrid Quad (what Ross did)

4) Spreadsheet: Geometry, Iterations

1) Potential Functions, Complex Analysis, Analytic Functions

Electromagnet Design of 11899 notes was simple since we had an infinite permeability flux return and knew that the flux density at the pole tip had a constant gradient to the zero-field line.

\[ \text{MMF } P = \phi = \mathcal{E} \, dA \]
\[ B = \frac{1}{2} B_{\phi t} \]
\[ P = \frac{1}{\Gamma_{\phi t}} \int \frac{1}{\Gamma_{\phi t}} dA = \int dA = L \int ds \]
\[ \text{MMF} = \frac{1}{2} B_{\phi t} \Gamma_{\phi t} \frac{1}{\omega_0} \]

We used MMF (Magnetic Motive Force) before to develop our force analogies, but just as in electrical circuits where EMF (Electromotive Force) is also known as the potential, we can say that the MMF is the magnetic potential \( V_0 \).

\[ \text{MMF} = V_0 = \frac{1}{2} B_{\phi t} \Gamma_{\phi t} \frac{1}{\omega_0} \]

\[ \tilde{V}_0 = 4_0 V_0 \]
Hybrid PM's are more complicated since there is not a simple $q = \infty$ flux return with a single air gap. There are several $q = 1$ gaps and PM sources. We need to start bookkeeping on flux. So we need a new approach. Lumped circuit models won't work.

From RS notes:

MAXWELL:

**INTEGRAL FORMULATION**
(macroscopic scale)

\[
\phi \int H \cdot d\mathbf{s} = I(t) = \int \mathbf{J} \cdot d\mathbf{a} \quad \Rightarrow \quad \nabla \times \mathbf{H} = \mathbf{J}
\]

\[\text{Ampere's Law (}\nabla \cdot \mathbf{H} = \mathbf{J}\)]

\[\text{FARADAY's Law: } \nabla \times \mathbf{E} \cdot d\mathbf{s} = -\int \mathbf{B} \cdot d\mathbf{a} = -\Phi \quad \Rightarrow \quad -\nabla \times \mathbf{E} = \mathbf{B} \quad \Rightarrow \quad \nabla \cdot \mathbf{B} = 0 \Rightarrow \nabla \cdot \mathbf{B} = \rho_m = 0 \quad \Rightarrow \quad \text{Ampere's law (} \nabla \cdot \mathbf{B} \text{)}
\]

**DIFFERENTIAL FORMULATION**
(microscopic scale)

\[
\text{curl}(\mathbf{E}) = 0
\]

Scalar + Vector Potentials:

For $\mathbf{J} = 0 \Rightarrow \nabla \times \mathbf{A} = 0$

1) $\nabla \times \mathbf{H} = 0 \Rightarrow \mathbf{H} = -\nabla V$

2) $\nabla \cdot \mathbf{B} = 0 \Rightarrow \nabla \cdot \mathbf{A} = 0$

LAPLACE Eqn. for $\nabla V$

\[
\nabla^2 \mathbf{V} = 0 \quad \text{(Scalar potential does not obey Laplace's Eqn.)}
\]

\[
\mathbf{B} = \mathbf{H} = -\nabla \mathbf{V} ; \quad \nabla \cdot \mathbf{B} = 0 \Rightarrow \nabla \cdot \nabla \mathbf{V} = \nabla^2 \mathbf{V} = 0
\]
LAPLACE Eqn. for $\vec{A}$ in 2-D case where $\frac{\partial^2}{\partial z^2} = 0$:

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{A} = \nabla^2 \vec{A} = 0$$

here $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$; $\vec{A} = (0, 0, A)$

Complex analysis opens up new analytical tools (see attached from mathematical methods for physics).

So KH starts by defining complex variable $z$ and potential function, $F$:

**Theory of a function of a complex variable**

Let $F(z) = A(x, y) + i\tilde{V}(x, y)$ be an analytical function of the complex variable $z = x + iy$ with real part $A(x, y)$ and imaginary part $\tilde{V}(x, y)$.

**Cauchy-Riemann:**

$$\frac{\partial F}{\partial x} = \frac{\partial A}{\partial x} + i\frac{\partial \tilde{V}}{\partial x} = F_x \text{ i }$$

$$\frac{\partial F}{\partial y} = \frac{\partial A}{\partial y} + i\frac{\partial \tilde{V}}{\partial y} = F_y \text{ i }$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2} \text{ i }$$

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial y^2} (-1) \Rightarrow \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = \nabla^2 F = 0$$

$A + \tilde{V}$ here satisfy same equations that vector potential $A$ + scalar potential $\tilde{V}$ describing fields $B_x, B_y$ did.

So we have $F = A + i\tilde{V}$

(2-D)

Relationship between complex potential + magnetic field:
For complex analysis of functions, a complex function must be 'analytic' and satisfy:

1) \( \frac{dF(z_0)}{dz} \) exist at \( z_0 \) and points of interest around \( z_0 \)

2) \( F \) satisfies Cauchy-Riemann condition:

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]

vector potential

Scalar potential

For \( F = A + i B \), it can be shown that:

\[ B^* = i \frac{dF}{dz} \quad \text{complex conjugate} \]

Sometimes \( F \) uses the vector potential, \( A \)

(2D FEA does Laplace equation) or the scalar potential, \( V \), (as we will) or

Sometimes problem is broken into parts to take advantage of geometry to simplify analysis.

2) Direct/Indirect Field Method - PM representation as charge

You can represent PM's as charge sheets in addition to current sheets.

Although a magnetic monopole does not really exist in a physical sense, \( \nabla \cdot B = \rho_m = 0 \),

\[ \oint S \cdot B \, ds = \phi = 0 \]

\( S \) is mag. charge, or flux, or Gauss's Law

\( \rho_m \) is mag. charge volume

Maxwell Eq.

From RS Notes ->
For homogeneously magnetized:

1. $\nabla \times \vec{H_c} = 0$ everywhere except at surface $\Rightarrow$ current sheet
2. $\nabla \cdot \vec{B} = 0$ everywhere except at surface $\Rightarrow$ charge sheet

$$\vec{H_c} = \frac{l}{A_s} \frac{I}{l s} \frac{th_s}{\mu_0} \frac{\sigma}{th_s}$$

**PM** CURRENT SHEETS **CHARGE SHEETS**

$$- \nabla \cdot \vec{B} = \rho_{eq} = \frac{Q}{\text{Vol}} = \frac{\pm \vec{B}}{\text{th}_{e}} \Rightarrow Q = \rho_{eq} \cdot \text{th}_{e} \cdot A_e = \pm \vec{B} \cdot A_e$$

$$Q/\text{Vol} \text{Vol} \text{Q/\text{Area Area}}$$

Convenient to use $\sigma = Q/A_e = \rho_{eq} \cdot \text{th}_{e} = \pm \vec{B}$

$$\nabla \times \vec{H_c} = \vec{H}_c = \frac{I}{A_s} \frac{\vec{H}_c}{\text{th}_{s}} \Rightarrow I = \rho_{eq} \cdot \text{th}_{s} \cdot L = \pm \vec{H}_c \cdot L$$

$$I/\text{Area Area} \text{I/\text{Len. Length}}$$

Convenient to use $I' = I/\mu = \text{i..th..} = \pm \vec{H}$

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Hybrid

- Solve a real problem with steel, air
- Recognize two things about the steel pole:
  - Constant scalar potential $\psi = \infty$
  - Pole can store no charge, $\sigma$ Bds = 0
- Steps:
  a) Design pole so that $\mu$ gives desired field distribution
  b) Determine pole scalar potential, $V_0$
  c) Break flux in and out of pole into 'direct' and 'indirect' fields
Hybrid (IRON + PM) 3-D Theory

Represent IRON with \( \mu = \infty \); PM with \( \mu_{||}, \mu_{\perp}, \rho_{eq} \)

Let's start with a point charge \( Q \) + 2 iron surfaces:

We construct a solution that satisfies MAXWELL in space outside iron and has total flux \( \Phi_1 = 0 \) entering surface 1. This solution is a superposition of 2 solutions to MAXWELL outside iron:

\[
\text{surface 1} \quad V_0 \quad \Rightarrow \quad (a) \quad V_d(r_1) = 0 \quad \text{DIRECT FIELDS} \quad + \quad (b) \quad V_i(r_1) = V_0 \quad \text{INDIRECT FIELDS}
\]

\( (a) \) \( Q \neq 0 \); \( \forall \Omega V = V_d(r_1) = 0 \); \( V_d(r_1) \rightarrow \mathbf{H}_d \rightarrow \Phi_d = \int \mathbf{H}_d \cdot d \mathbf{A} = +Q \cdot C_1 \)

\( (b) \) \( Q = 0 \); \( \forall \Omega V = V_i(r_1) = V_0 \); \( V_i(r_1) \rightarrow \mathbf{H}_i \rightarrow \Phi_i = \int \mathbf{H}_i \cdot d \mathbf{A} = -V_0 \cdot C \)

\( (c) \) \( V = V_d + V_i \rightarrow \mathbf{H} = \mathbf{H}_d + \mathbf{H}_i \rightarrow \Phi = \Phi_d + \Phi_i = +Q \cdot C_1 - V_0 \cdot C_2 = \)

\[ \Rightarrow \quad \mathbf{V}_{\text{SURFACE} 1} = \mathbf{V}_0 = Q/C_2 \]
As functions of geometry.

D) Develop expressions for direct & indirect flux with help of Schwartz-Christoffel transforms. (Ross did this for us)

Express direct flux as flux per unit length, \( \Phi_d \), and indirect flux as flux per unit length - pole scalar potential.

\[ \Phi_i = \Phi_i' / V_0 \]

\[ \text{Check} \quad \frac{\Phi_i}{V_0} = \frac{\sum \Phi_0'}{\sum \Phi_i'} \]

Yes - you're done

No - change geometry

Iterate

When finding direct fields, \( \Phi_d \), all potentials are set to zero. The magnetic charge density will split flux based on \( c_1 \) and \( c_2 \) (capacitances).

When finding indirect fields set pole potential to \( V_0 \), symmetry plane potentials to zero, outer flux return? (? Ask Ross). Set all changes to zero.

**Calculation of \( c_1 \)**: \( c_1 \) is just the fraction of \( Q \) that goes into surface 1 when both surfaces are on zero scalar potential.

Result is intuitive:

\[ c_1 = \frac{V_1(f_0)}{V_0} \]

\[ c_1 = \frac{\frac{V}{0}}{\cdot Q} \]

\[ c_1 = \frac{3}{3} \]

\[ V=0 \]
Excess Flux Concept

- Need to account for flux in regions where flux is not uniform or is unknown leakage flux from design field.

- Introduce dimensionless coefficient, $E$, to account for this flux.

$$\phi = \nu E \quad (\text{in our problem})$$

- Use FEA model or Schwarz-Cristoffel transformation (Ross has done this).

Schwarz-Cristoffel Transformation:

- What it is: A procedure to derive a transformation that maps the interior of a polygon to a half-plane.
- What it is good for: Establishing closed-form relationships between two relevant complex quantities that are analytical functions of each other (like $\gamma$, $F$, $B^*$).

How we are going to use it: To get closed-form expressions for excess flux coefficients.

S-C Recipe:

- upper half $t$-plane

  interior of polygon in $z$-plane mapped to upper half $t$-plane

  corners of polygon in $z$-plane numbered sequentially and mapped on sequentially numbered points on the real axis of the $t$-plane with

$$\frac{dz}{dt} = c_i \frac{T(c_i) (t-t_i)^{-ki/\pi}}{i}$$

- Different $c_i$, $c_2$
1) Input Geometry:

\( g_1, g_2, g_4, g_6, g_9, WPM, E_1, L \)

2) Input PM: \( B_r \approx 1.2T \) NdFeB, 0.4T Ferrite

- choose \( q_2 \) for narrowest pole tip that still gives adequate field quality
- choose \( \alpha + \beta_y \) to minimize PM volume,
- choose \( h_3 \) as close to \( g_2 \) as possible, \( h_3 = g_2 \)
- choose \( g_9 + 1 \to h_2 \) as possible

\[ \sum_{k=1}^{2} \hat{\Phi}_0^k = \sum_{l=1}^{9} \hat{\Phi}_i^l, \quad \text{where} \quad \hat{\Phi}_0^k = \frac{\Phi_0}{B_r} + \frac{E_1}{L} \]

\[ \hat{\Phi}_0^1 = w_{PM}; \quad \hat{\Phi}_0^2 = \frac{\Phi_0^2}{2h_2}; \quad \text{where} \quad \Phi_0 \equiv \Phi_0 / B_r \]

\[ \hat{\Phi}_i^1 = Re \{ \hat{\Phi}_2^1 \}; \quad \hat{\Phi}_i^2 = \frac{(x_1)}{2h_2}; \quad \text{where} \quad \hat{\Phi}_i^1 = \hat{\Phi}_i / V_0 + \hat{\Phi}_i = \frac{V_0}{X_0} \]

\[ \hat{\Phi}_i^3 = (x_6 - x_4) / y_4; \quad \hat{\Phi}_i^4 = \frac{r_2}{2h_2} (\frac{x_6}{X_4}) \]

\[ \hat{\Phi}_i^5 \approx \text{negligible}; \quad \hat{\Phi}_i^6 = \frac{1}{2} \left[ \ln \left( \frac{a + e}{y_6} \right) + 2a \ln \left( \frac{a}{y_6} \right) \right] \]

\[ \hat{\Phi}_i^7 = \frac{1}{2} \int_0^1 \frac{\partial}{\partial \theta} \frac{\partial}{\partial a} \left( \frac{a^2}{\theta} \right) \left( \frac{\partial}{\partial a} \right) \ln \frac{a e}{\theta} \]

where \( a_{END} = \frac{\pi r^2}{8} \left\{ (x_0 - x_4)(y_4 + x_2(y_2^2 - y_0^2)) + \frac{r_2^2}{2} \left( \frac{1}{2} + \ln \frac{x_0}{x_6} \right) \right\} \)

\[ \hat{\Phi}_i^1 \to \text{from first principles} \]

\[ \hat{\Phi}_i^2, 3, 4, 6, 8 \to \text{from Schwartz-Cristoffel} \]

\[ I_{3, 7} \to \text{uniform flux} \]

\[ \hat{I}_d \to \text{Id = Br x area} \quad \rightarrow \hat{I}_d^' = \hat{I}_d / (Br x \text{area}) \]

\[ V_0 = \frac{\sum_k \hat{\Phi}_0^k}{\sum_k \hat{\Phi}_i^k}; \quad + \quad V_0 = B_0 r_1 / 2 \]